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Mathematical and Numerical Modeling of Two-Phase Compressible Flows with Micro-Inertia

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A new model with full coupling between micro- and macroscale motion is developed for compressible multiphase mixtures. The equations of motion and the coupling microstructural equation (an analogue of the Rayleigh–Lamb equation) are obtained by using the Hamilton principle of stationary action. In the particular case of bubbly fluids, the resulting model contains eight partial differential equations (one-dimensional case) and is unconditionally hyperbolic. The equations are solved numerically by an adapted Godunov method. The model and methods are validated for two very different test problems. The first one consists of a wave propagating in a liquid containing a small quantity of gas bubbles. Computed oscillating shock waves fit perfectly the experimental data. Then the one-dimensional multiphase model is used as a reduction tool for the multidimensional interaction of a shock wave with a large bubble. Good agreement is again obtained. © 2002 Elsevier Science

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INTRODUCTION

A typical example of a fluid with micro-inertia is a fluid containing gas bubbles. Since the pioneering works of Iordansky [21], Kogarko [26], and van Wijngaarden [45], a number of mathematical models of bubbly fluids have been proposed. At least two essentially different methods were used to derive the governing equations. The first is based on averaging of the local instantaneous conservation laws; see [7, 22, 30], among others. The second, the variational approach, uses Hamilton's principle of stationary action; see [3, 4, 8, 13, 14] and others. The advantage of the variational method is that the knowledge of only one scalar function written in terms of the average variables, the Lagrangian of the system, allows us

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to obtain a closed system of governing equations. However, it is not an easy task to find an explicit form of the Lagrangian in terms of the average variables. To do so, a number of hypotheses should be developed.

By using the variational approach, Berdichevsky [4] and Bedford and Drumheller [3] obtained the two-fluid equations in the case of barotropic components. Geurst [13, 14] considered the case when one of the components is incompressible. By using the Hamilton principle, he obtained the governing equations and studied their hyperbolicity. Pauchon and Smereka [31] compared the averaging and the variational approach in the case of incompressible dispersed flows. They found that in the dilute limit, the equations of motion obtained by both methods are the same. In the one-dimensional case, they proposed a criteria of hyperbolicity of the governing equations. In all these papers, the method of Lagrange multipliers was used to take into account the constraints expressing the conservation of mass and the volume fraction constraint (the sum of the volume fractions must equal to one). A different approach was used by Gavrilyuk *et al.* [11], Gavrilyuk and Gouin [12], and Gouin and Gavrilyuk [19] for the case in which one of the components is incompressible.

They integrated differential constraints in Lagrange coordinates and found the variation of Hamilton's action directly in terms of virtual displacements. This permits one to find simultaneously the governing equations (which do not form a system of conservation laws) and a complete set of Rankine–Hugoniot conditions. The fact that the knowledge of the Lagrangian of the system gives automatically weak formulation of the governing equations is well known for one-component flows; see [18, 37, 39] and others. A simple criteria of hyperbolicity was also proposed: the convexity of the internal energy provides the hyperbolicity of the governing equations.

In this paper we extend the approach "without Lagrange multipliers" to the case of dispersed flow of two compressible components when each of the components may have its own temperature. This case is very important for a variety of applications: Richtmyer–Meshkov instability, deflagration-to-detonation transition in porous energetic materials, etc. Among the models describing an immiscible mixture of two compressible fluids, the model of Baer and Nunziato [2] (henceforth referred to as the BN model) has to be mentioned. A variant of the BN model has also been proposed by Saurel and Abgrall [36]. The BN model consists of the mass conservation laws for each component and nonconservative momentum and energy equations. This model is closed by a transport equation for the volume concentration φ, which is an independent variable in the case of compressible fluids. The model is hyperbolic unlike the one based on the assumption of "equal pressure" (see, for example, a discussion on the "equal pressure" model in Stewart and Wendroff [42]). However, the BN model should be generalized in order to describe correctly phenomena with inertia effects (for example, the flow features being exhibited during the shock wave propagation in bubbly fluids).

To obtain a set of governing equations describing the phenomena with internal inertia, we use an extended Hamilton's principle with a Lagrangian which is not limited to the case of small volume fraction of gas bubbles. The system is closed by a second-order differential equation for φ obtained by varying the Hamilton action with respect to φ. This equation is an analogue of the Rayleigh–Lamb equation describing the bubble pulsations in an incompressible fluid; see [27, 33] and others. We prove the sufficient criterion of hyperbolicity of the system: it is hyperbolic if the total energy is convex. The model is specified for the case when the added mass term is negligible compared with the inertia term. In contrast to the BN model, new terms related to the inertia effects come also into the

momentum and energy equations for the continuous phase. The governing equations are presented in a form suitable for numerical calculations. In particular, the equation for the volume concentration can be rewritten in terms of the Riemann invariants. For this system we can prove the unconditional hyperbolicity: the equations are hyperbolic apart from a set of parameters of the zero measure. Finally, dissipative algebraic terms compatible with the second law of thermodynamics are introduced in the model.

A numerical method proposed by Saurel and Abgrall [36] based on the Godunov method (1979) is then adapted to this model and provides an efficient resolution scheme. The ability of the model to solve correctly various physical problems is demonstrated for the two following cases:

- the shock wave propagation in a fluid containing small gas bubbles (bubbly fluid),
- the shock wave interaction with the material interfaces.

In the first case, a very good agreement between numerical results and experimental results of Kameda *et al.* [23] is demonstrated. In the second case, the one-dimensional two-fluid model was used as a reduction tool for the two-dimensional problem of interaction of a shock wave with a large bubble. Good agreement between the 1D and the averaged 2D results is again obtained.

1. GENERAL APPROACH BASED ON HAMILTON'S PRINCIPLE OF STATIONARY ACTION

The Lagrangian formulation of classical mechanics is based on the Hamilton variational principle. This principle states that the real motion of a system is the extremal of the Hamilton action in a class of trajectories joining two fixed points in coordinate space. The Hamilton action is defined as an integral of the Lagrangian over time. In general, the Lagrangian of the system is the difference between kinetic and potential energies. Formulation of the Hamilton principle for the continuum mechanics, which is an infinite-dimensional system, can be found in Serrin [40], Lin [28], Seliger and Whitham [38], Sedov [37], and others. An extension of the Hamilton principle for the two-fluid hydrodynamics is formulated in Berdichevsky [4], Bedford and Drumheller [3], Geurst [13, 14], Pauchon and Smereka [31], etc. In this section we extend the variational approach to the case of two nonbarotropic compressible components.

1.1. Basic Notations

We consider a mixture of two immiscible components. Each α-th constituent has its own averaged characteristics: the local velocity $\mathbf{u}_\alpha$, the local density ρ_α^0, the partial density ρ_α, the volume fraction φ_α, the local entropy per unit mass η_α, the local internal energy per unit mass $\varepsilon_\alpha(\rho_\alpha^0, \eta_\alpha)$, and the local temperature θ_α, $\alpha = 1, 2$. The partial densities ρ_α are defined by the formulae

$$\rho_\alpha = \varphi_\alpha \rho_\alpha^0. \tag{1.1}$$

They obey the mass conservation law

$$\frac{\partial}{\partial t}\rho_\alpha + \operatorname{div}(\rho_\alpha \mathbf{u}_\alpha) = 0. \tag{1.2}$$

For dissipation-free motions the local entropies conserve along the trajectories

$$\frac{d_\alpha}{dt}\eta_\alpha = 0 \tag{1.3}$$

with

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \nabla.$$

The partial volume entropies S_α defined by the formulae

$$S_\alpha = \rho_\alpha \eta_\alpha \tag{1.4}$$

satisfy for continuous motions the entropy conservation law

$$\frac{\partial}{\partial t} S_\alpha + \operatorname{div}(S_\alpha \mathbf{u}_\alpha) = 0. \tag{1.5}$$

We suppose that the volume fractions φ_α satisfy the saturation condition

$$\varphi_1 + \varphi_2 = 1. \tag{1.6}$$

We shall use the subscript "1" and "2" for the continuous and dispersed phase, respectively, and shall often set:

$$\varphi_2 = \varphi, \quad \varphi_1 = 1 - \varphi. \tag{1.7}$$

We also suppose that the Gibbs identity for each component is satisfied

$$\theta_\alpha d\eta_\alpha = d\varepsilon_\alpha + p_\alpha d\left(\frac{1}{\rho_\alpha^0}\right). \tag{1.8}$$

Here $p_\alpha(\rho_\alpha^0, \eta_\alpha)$ are local average pressures.

1.2. Total Energy of the System

To use the Hamilton principle of stationary action, we need an expression of the total energy of mixture. We propose to take the total energy per unit volume in the form:

$$E = \underbrace{\sum_{\alpha=1}^{2} \rho_\alpha \frac{|\mathbf{u}_\alpha|^2}{2}}_{(1)} + \underbrace{\frac{m}{2}\left(\frac{d_i\varphi}{dt}\right)^2}_{(2)} + \underbrace{\sum_{\alpha=1}^{2} \rho_\alpha \varepsilon_\alpha\left(\frac{\rho_\alpha}{\varphi_\alpha}, \frac{S_\alpha}{\rho_\alpha}\right)}_{(3)}$$
$$+ \underbrace{(\rho_1 + \rho_2)e(\varphi)}_{(4)} + \underbrace{\frac{k}{2}|\nabla\varphi|^2}_{(5)} + \underbrace{\frac{d}{2}|\mathbf{u}_2 - \mathbf{u}_1|^2}_{(6)}. \tag{1.9}$$

In the analysis that follows in this section, we shall not use this particular form of the total energy to derive the governing equations. They will be derived in the general case. However, to do numerical calculations, this explicit expression will be needed in the following sections. Let us precisely define each term in (1.9):

(1) represents the kinetic energy of translational motion;

(2) is the pulsation kinetic energy,

$$\frac{d_i}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_i \nabla$$

(the subscript "i" means "interface"); the expression for the interface velocity $\mathbf{u}_i$ should be given *a priori*. The choice of $\mathbf{u}_i$ is very important because it defines the structure of governing equations (see a discussion at the end of Section 2).

(3) is the internal energy per unit volume;

(4) corresponds to the potential energy related to the internal structure (the "configuration energy"; Passman *et al.* [32]). The function e should be given *a priori*;

(5) takes into account the energy due to the macroscopic nonhomogeneity of the mixture. The volume fraction φ plays the role of order parameter as in the Cahn–Hillard approach (1957);

(6) is the kinetic energy related to the added mass effect.

In a particular case of dilute suspension of gas bubbles of the same size, the following classical expressions for the coefficients m and d could be given:

$$m \approx \frac{\rho_1^0}{3}\left(\frac{3}{4\pi N}\right)^{2/3}\varphi^{-1/3}, \quad d \approx \rho_1^0 \frac{\varphi}{2}.$$

Here N is the number of bubbles per unit volume which is supposed to be constant. This is a quite good approximation for the problems which will be considered in Section 5. The formulae for m and d are justified as follows. Consider the bubble of radius $R(t)$ moving with the velocity $\mathbf{u}_2$ in an incompressible perfect fluid having the velocity $\mathbf{u}_1$ at infinity. The kinetic energy of the fluid resulting from the relative motion of the bubble and its radial pulsations is (see, for example, Lamb, [27])

$$T_f = \frac{\pi R^3 \rho_1^0}{2}\left(4\left(\frac{dR}{dt}\right)^2 + \frac{2}{3}|\mathbf{u}_2 - \mathbf{u}_1|^2\right).$$

Then for a dilute bubbly fluid, when we neglect the bubble interaction, the kinetic energy T_f^c of the fluid per unit volume in the presence of N bubbles in the continuum limit (when all the unknown functions $R, \mathbf{u}_1, \mathbf{u}_2$ become functions of $(t, \mathbf{x})$) will be

$$T_f^c = \frac{\pi R^3 \rho_1^0 N}{2}\left(4\left(\frac{d_i R}{dt}\right)^2 + \frac{2}{3}|\mathbf{u}_2 - \mathbf{u}_1|^2\right).$$

The definition of the volume fraction $\varphi_2 = \varphi = \frac{4}{3}\pi R^3 N$ permits us to rewrite the final expression for the volume kinetic energy T_f^c in the form

$$T_f^c = \frac{m(\varphi)}{2}\left(\frac{d_i \varphi}{dt}\right)^2 + \frac{d(\varphi)}{2}|\mathbf{u}_2 - \mathbf{u}_1|^2,$$

with $m(\varphi)$ and $d(\varphi)$ defined above. This formula is a good approximation even for the case of compressible fluid component. In the following, we choose

$$\frac{d_i}{dt} = \frac{d_1}{dt}$$

(see the discussion at the end of Section 2).

The expression for the capillary term (5) could be given as follows. We take the capillary energy in the form $c|\nabla\rho|^2/2$, where ρ is the average density $\rho = \rho_1 + \rho_2$, and $c = const$ is the capillary coefficient [34]. By supposing that the average local densities are approximately constants, we get $c|\nabla\rho|^2/2 \approx c(\rho_1^0 - \rho_2^0)^2|\nabla\varphi|^2/2$. Hence,

$$k \approx c\left(\rho_1^0 - \rho_2^0\right)^2.$$

1.3. Lagrangian of the System

The Lagrangian of the system is taken in the usual form $L = T - U$, where T is the kinetic energy and U is the potential energy. Definition (1.9) of the total energy allows us to separate E into the kinetic and the potential energy

$$\begin{aligned} T &= \sum_{\alpha=1}^{2} \rho_\alpha \frac{|\mathbf{u}_\alpha|^2}{2} + \frac{m}{2}\left(\frac{d_i\varphi}{dt}\right)^2 + \frac{d}{2}|\mathbf{u}_2 - \mathbf{u}_1|^2, \\ U &= \sum_{\alpha=1}^{2} \rho_\alpha \varepsilon_\alpha\left(\frac{\rho_\alpha}{\varphi_\alpha}, \eta_\alpha\right) + \rho e(\varphi) + \frac{k}{2}|\nabla\varphi|^2. \end{aligned} \tag{1.10}$$

If the translational derivative

$$\frac{d_i}{dt}$$

is Galilean invariant, this form of the Lagrangian guarantees Galilean invariance of the governing equations. The variables $\mathbf{u}_\alpha$ and η_α are not very convenient for calculations. We introduce new variables: the partial momentum of each component $\mathbf{j}_\alpha = \rho_\alpha\mathbf{u}_\alpha$ and the partial entropy S_α. Consider the Lagrangian in a general form

$$L = L\left(\mathbf{j}_1, \mathbf{j}_2, \rho_1, \rho_2, S_1, S_2, \varphi, \frac{\partial\varphi}{\partial t}, \nabla\varphi\right). \tag{1.11}$$

We shall see later that the total energy (1.9) is the Legendre transformation of L

$$E = \sum_{\alpha=1}^{2} \frac{\partial L}{\partial \mathbf{j}_\alpha}\mathbf{j}_\alpha + \frac{\partial\varphi}{\partial t}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} - L.$$

1.4. Virtual Motions and Variation of Dependent Variables

Let $\mathbf{X}_\alpha$ be Lagrangian coordinates of α-th component. The motion of the mixture is given by two smooth maps $\mathbf{x} = \boldsymbol{\varphi}_\alpha(\mathbf{X}_\alpha, t)$. Let λ_1, λ_2 be real numbers in the vicinity of zero. Let us consider two families of virtual motions $\mathbf{x} = \boldsymbol{\Phi}_\alpha(\mathbf{X}_\alpha, t, \lambda_\alpha)$ such that $\boldsymbol{\Phi}_\alpha(\mathbf{X}_\alpha, t, 0) = \boldsymbol{\varphi}_\alpha(\mathbf{X}_\alpha, t)$. The virtual displacements corresponding to each family are defined by

$$\boldsymbol{\zeta}_\alpha = \frac{\partial\boldsymbol{\Phi}_\alpha}{\partial\lambda_\alpha}(\mathbf{X}_\alpha, t, \lambda_\alpha)\bigg|_{\lambda_\alpha=0}.$$

The virtual displacements $\boldsymbol{\zeta}_\alpha$ are defined as functions of the Lagrangian coordinates. In the following we consider them as functions of Eulerian coordinates by using the inverse map $\mathbf{X}_\alpha = \boldsymbol{\varphi}_\alpha^{-1}(\mathbf{x}, t)$. The variations of the dependent variables for fixed values of Eulerian

coordinates, corresponding to each family of virtual motions, are given by the formulae (see for details Gavrilyuk *et al.* [11], Gavrilyuk and Gouin [12]):

$$\delta_\alpha \rho_\alpha = -div(\rho_\alpha \boldsymbol{\zeta}_\alpha) \tag{1.12}$$

$$\delta_\alpha \boldsymbol{j}_\alpha = \frac{\partial}{\partial t}(\rho_\alpha \boldsymbol{\zeta}_\alpha) + div(\mathbf{j}_\alpha \otimes \boldsymbol{\zeta}_\alpha - \boldsymbol{\zeta}_\alpha \otimes \boldsymbol{j}_\alpha) \tag{1.13}$$

$$\delta_\alpha S_\alpha = -div(S_\alpha \boldsymbol{\zeta}_\alpha). \tag{1.14}$$

Here the divergence of a linear transformation A is defined as follows: $div(A\mathbf{h}) = div(A)\mathbf{h}$ for any constant vector $\mathbf{h}$. In particular,

$$div(\mathbf{a} \otimes \mathbf{b}) = \mathbf{b}\, div\, \mathbf{a} + \frac{\partial \mathbf{b}}{\partial \mathbf{x}}\mathbf{a}$$

for any vector fields $\mathbf{a}$ and $\mathbf{b}$.

The two families $\mathbf{x} = \boldsymbol{\Phi}_\alpha(\mathbf{X}_\alpha, t, \lambda_\alpha)$ are independent. They do not change the corresponding volume fraction, which is an internal parameter of the mixture. The variation of the Lagrangian through each family gives the equations of motion of each component. Finally, we need a third independent family that permits us to vary only the volume fraction φ. It can be chosen in the form $\varphi = \hat{\varphi}(\mathbf{x}, t, \lambda)$. The other parameters of the mixture are unchanged. The variation of the volume fraction φ for this family will be denoted by $\delta\varphi$:

$$\delta\varphi = \left.\frac{\partial \hat{\varphi}(\mathbf{x}, t, \lambda)}{\partial \lambda}\right|_{\lambda=0}.$$

1.5. Euler–Lagrange Equations

Let $B \times [t_1, t_2]$ be a domain in the space–time. The Hamilton action

$$a = \int_{t_1}^{t_2} dt \int_B L\, d\mathbf{x}$$

submitted to the three one-parameter families becomes the function of $\lambda_\alpha, \alpha = 1, 2$ or λ, respectively. We denote:

$$\delta_\alpha a = \left.\frac{da}{d\lambda_\alpha}\right|_{\lambda_\alpha=0}, \quad \alpha = 1, 2 \quad \text{and} \quad \delta a = \left.\frac{da}{d\lambda}\right|_{\lambda=0}.$$

The equalities $\delta_\alpha a = 0$ and $\delta a = 0$ give the momentum equation for each component, and the microstructural equation for the volume fraction φ, respectively. We shall suppose that the variations $\boldsymbol{\zeta}_\alpha$ and $\delta\varphi$ vanish on the boundary $\partial(B \times [t_1, t_2])$. For the families $\boldsymbol{\Phi}_\alpha$, the corresponding variations of the Hamilton action are

$$\delta_\alpha a = \int_{t_1}^{t_2} dt \int_B d\mathbf{x} \left\{ \frac{\partial L}{\partial \boldsymbol{j}_\alpha}\delta_\alpha \mathbf{j}_\alpha + \frac{\partial L}{\partial \rho_\alpha}\delta_\alpha \rho_\alpha + \frac{\partial L}{\partial S_\alpha}\delta_\alpha S_\alpha \right\}. \tag{1.15}$$

Let us denote

$$\mathbf{K}_\alpha = \frac{\partial L}{\partial \mathbf{j}_\alpha}, \quad R_\alpha = \frac{\partial L}{\partial \rho_\alpha}, \quad \theta_\alpha = -\frac{\partial L}{\partial S_\alpha}. \tag{1.16}$$

The sign "minus" in the definition of temperature θ_α is chosen in accordance with Gibbs identity (1.8). Taking into account definitions (1.16) and formulae (1.12)–(1.14), we get from (1.15):

$$\delta_\alpha a = \int_{t_1}^{t_2} dt \int_B d\mathbf{x} \Big\{ \mathbf{K}_\alpha \left(\frac{\partial}{\partial t}(\rho_\alpha \boldsymbol{\zeta}_\alpha) + div(\mathbf{j}_\alpha \otimes \boldsymbol{\zeta}_\alpha - \boldsymbol{\zeta}_\alpha \otimes \mathbf{j}_\alpha) \right) - R_\alpha div(\rho_\alpha \boldsymbol{\zeta}_\alpha) + \theta_\alpha div(S_\alpha \boldsymbol{\zeta}_\alpha) \Big\}.$$

Since the virtual displacements $\boldsymbol{\zeta}_\alpha$ vanish on the boundary $\partial(B \times [t_1, t_2])$, we obtain after using the Gauss–Green–Ostrogradsky theorem

$$\delta_\alpha a = \int_{t_1}^{t_2} dt \int_B d\mathbf{x} \Big\{ -\rho_\alpha \boldsymbol{\zeta}_\alpha \frac{\partial \mathbf{K}_\alpha}{\partial t} - \left(\boldsymbol{\zeta}_\alpha \left(\frac{\partial \mathbf{K}_\alpha}{\partial \mathbf{x}} \right) \mathbf{j}_\alpha - \mathbf{j}_\alpha \left(\frac{\partial \mathbf{K}_\alpha}{\partial \mathbf{x}} \right) \boldsymbol{\zeta}_\alpha \right) + \rho_\alpha \boldsymbol{\zeta}_\alpha \nabla R_\alpha - S_\alpha \boldsymbol{\zeta}_\alpha \nabla \theta_\alpha \Big\}.$$

It can be written as

$$\delta_\alpha a = \int_{t_1}^{t_2} dt \int_B d\mathbf{x} \Big\{ -\rho_\alpha \frac{\partial \mathbf{K}_\alpha}{\partial t} - rot\, \mathbf{K}_\alpha \times \mathbf{j}_\alpha + \rho_\alpha \nabla R_\alpha - S_\alpha \nabla \theta_\alpha \Big\} \boldsymbol{\zeta}_\alpha.$$

Hence, the momentum equations are

$$\rho_\alpha \frac{\partial \mathbf{K}_\alpha}{\partial t} + rot\, \mathbf{K}_\alpha \times \mathbf{j}_\alpha - \rho_\alpha \nabla R_\alpha + S_\alpha \nabla \theta_\alpha = 0, \quad \alpha = 1, 2. \tag{1.17}$$

Now, let us calculate the variation corresponding to the third family

$$\delta a = \int_{t_1}^{t_2} dt \int_B d\mathbf{x} \left\{ \frac{\partial L}{\partial \varphi} \delta\varphi + \frac{\partial L}{\partial \left(\frac{\partial \varphi}{\partial t} \right)} \delta \left(\frac{\partial \varphi}{\partial t} \right) + \frac{\partial L}{\partial (\nabla \varphi)} \delta(\nabla \varphi) \right\}.$$

Since

$$\delta \left(\frac{\partial \varphi}{\partial t} \right) = \frac{\partial (\delta \varphi)}{\partial t}, \quad \delta(\nabla \varphi) = \nabla(\delta \varphi),$$

we get by integration by parts and by using the fact that $\delta\varphi$ vanishes on the boundary the following equation for φ:

$$\frac{\delta L}{\delta \varphi} \equiv \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \left(\frac{\partial \varphi}{\partial t} \right)} \right) - div \left(\frac{\partial L}{\partial (\nabla \varphi)} \right) = 0. \tag{1.18}$$

The microstructural equation (1.18) is the analogue of the Rayleigh–Lamb equation describing the bubble which expands and contracts in surrounding liquid; see [27, 33] and others.

1.6. Conservation Laws and Hyperbolicity Criterion

The Noether theorem says that the governing equations (1.2), (1.5), (1.17), and (1.18) admit the conservation of total momentum and total energy corresponding to the invariance of the Lagrangian with respect to space-and-time shifts. These conservation laws can be obtained by straightforward calculations. The total momentum conservation law is

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{2}\rho_\alpha \mathbf{K}_\alpha - \nabla\varphi\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}\right) + div\left(\sum_{\alpha=1}^{2}\left(\mathbf{j}_\alpha\otimes\mathbf{K}_\alpha - \left(\mathbf{j}_\alpha\mathbf{K}_\alpha + \rho_\alpha\frac{\partial L}{\partial\rho_\alpha} + S_\alpha\frac{\partial L}{\partial S_\alpha}\right)I\right) - \frac{\partial L}{\partial(\nabla\varphi)}\otimes\nabla\varphi + LI\right) = 0.$$

Here I is the identity tensor. The total energy conservation law is

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{2}\mathbf{j}_\alpha\mathbf{K}_\alpha + \frac{\partial\varphi}{\partial t}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} - L\right) + div\left(\sum_{\alpha=1}^{2} -R_\alpha\mathbf{j}_\alpha + S_\alpha\theta_\alpha\mathbf{u}_\alpha + \frac{\partial\varphi}{\partial t}\frac{\partial L}{\partial(\nabla\varphi)}\right) = 0.$$

In general, there are no additional conservation laws in terms of dependent variables. Hence, the governing equations cannot be written in divergence form. This does not allow us to obtain a sufficient number of Rankine–Hugoniot conditions by using the classical approach based on the divergence form of equations. In the papers by Gavrilyuk *et al.* [11] and Gouin and Gavrilyuk [19], it was shown (in the case when one of the phases is incompressible) that Hamilton's principle gives a sufficient set of Rankine–Hugoniot conditions associated with the nonconservative form of governing equations. The aim of the present paper does not include the generalization of this result to the case of compressible mixtures. In this section we focus on the question of hyperbolicity.

Of course, we are not able to calculate the eigenvalues in explicit form for the general case. To prove the hyperbolicity, the only possibility is to rewrite governing equations in symmetric form of Friedrichs. But the method, which permits us to rewrite automatically governing equations in symmetric form, is also based on the divergence form of the governing equations (Godunov [16] and Friedrichs and Lax [10]). This is why we consider a restrictive class of *potential* flows, which admits a divergence form. Of course this divergence form does not necessarily imply physical shock conditions.

The following definition of potential flows is given in Gavrilyuk *et al.* [11] and Gavrilyuk and Gouin [12]. It generalizes the definition of potential flows proposed by Geurst [13] (see also Wallis [46]) for a particular case of a mixture with one incompressible component.

DEFINITION. The flow of the α-th component is called *potential* if $rot\,\mathbf{K}_\alpha = 0$, $\eta_\alpha = const$.

Obviously, all the one-dimensional isentropic flows are potential. The equations of potential motions are

$$\frac{\partial}{\partial t}\rho_\alpha + div\,\mathbf{j}_\alpha = 0,\quad \frac{\partial}{\partial t}\mathbf{K}_\alpha - \nabla\left(\frac{\partial L}{\partial\rho_\alpha}\right) = 0,\quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}\right) + div\left(\frac{\partial L}{\partial(\nabla\varphi)}\right) - \frac{\partial L}{\partial\varphi} = 0. \tag{1.19}$$

Here

$$L = L\left(\rho_\alpha, \mathbf{j}_\alpha, \varphi, \frac{\partial\varphi}{\partial t}, \nabla\varphi\right)$$

is the Lagrangian and

$$\mathbf{K}_\alpha = \frac{\partial L}{\partial \mathbf{j}_\alpha}.$$

System (1.19) is a system of conservation laws, which admits the energy conservation law with the energy density

$$E = \sum_{\alpha=1}^{2} \frac{\partial L}{\partial \mathbf{j}_\alpha}\mathbf{j}_\alpha + \frac{\partial\varphi}{\partial t}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} - L \equiv \sum_{\alpha=1}^{2} \mathbf{K}_\alpha\mathbf{j}_\alpha + \frac{\partial\varphi}{\partial t}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} - L. \tag{1.20}$$

It is possible to give a sufficient criterion of hyperbolicity of system (1.19) by using the Godunov–Friedrichs–Lax approach (Godunov [16] and Friedrichs and Lax [10]). For this, we have to rewrite the last equation of (1.19) as a conservative system of two first-order quasi-linear equations. Let

$$b = \frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}, \quad \mathbf{w} = \nabla\varphi.$$

We consider the energy E as function of the variables ρ_α, $\mathbf{K}_\alpha$, φ, b and $\mathbf{w}$. Formula (1.20) implies

$$\frac{\partial E}{\partial\rho_\alpha} = -\frac{\partial L}{\partial\rho_\alpha}, \quad \frac{\partial E}{\partial\mathbf{K}_\alpha} = \mathbf{j}_\alpha, \quad \frac{\partial E}{\partial\varphi} = -\frac{\partial L}{\partial\varphi}, \quad \frac{\partial E}{\partial\mathbf{w}} = -\frac{\partial L}{\partial\mathbf{w}}, \quad \frac{\partial\varphi}{\partial t} = \frac{\partial E}{\partial b}.$$

In these variables the equations of motion are written as

$$\begin{gathered}
\frac{\partial\rho_\alpha}{\partial t} + div\left(\frac{\partial E}{\partial\mathbf{K}_\alpha}\right) = 0,\\
\frac{\partial\mathbf{K}_\alpha}{\partial t} + \nabla\left(\frac{\partial E}{\partial\rho_\alpha}\right) = 0,\\
\frac{\partial\varphi}{\partial t} = \frac{\partial E}{\partial b},\\
\frac{\partial b}{\partial t} - div\left(\frac{\partial E}{\partial\mathbf{w}}\right) + \frac{\partial E}{\partial\varphi} = 0,\\
\frac{\partial\mathbf{w}}{\partial t} - \nabla\left(\frac{\partial E}{\partial b}\right) = 0.
\end{gathered} \tag{1.21}$$

If we add the following constraint on the initial conditions $rot\,\mathbf{K}_\alpha|_{t=0} = 0$, $(\mathbf{w} - \nabla\varphi)|_{t=0} = 0$, the system (1.21) will be equivalent to (1.19). We can now formulate the following theorem.

THEOREM 1. *If the energy* $E(\rho_\alpha, \mathbf{K}_\alpha, \varphi, b, \mathbf{w})$ *defined by* (1.20) *is convex, the system* (1.21) *is hyperbolic. Moreover, the system can be rewritten as a symmetric t-hyperbolic system in the sense of Friedrichs.*

The proof of the theorem is a direct application of the result of Godunov–Friedrichs–Lax to the system (1.21). This sufficient criterion is quite restrictive. In particular, it cannot be applied if the pulsation energy is taken in the form (2) and the capillary energy (5) is equal to zero (see the expression of the total energy (1.9)). Indeed, without the capillary energy, the term

$$\frac{m}{2}\left(\frac{d_i\varphi}{dt}\right)^2$$

is degenerate with respect to b and $\mathbf{w}$. However, other possible forms of the pulsation energy may exist, especially for not-small volume fractions of dispersed phase. For example, the pulsation energy can be taken in the form

$$\sum_{\alpha=1}^{2}\frac{m_\alpha}{2}\left(\frac{d_\alpha\varphi}{dt}\right)^2. \tag{1.22}$$

This energy is convex with respect to b and $\mathbf{w}$ if the velocities $\mathbf{u}_\alpha$ are different. When Theorem 1 cannot be applied, direct calculations may be done for a given form of the total energy of the system. We shall do it in the next section for the case of bubbly fluids.

2. APPLICATION TO BUBBLY FLUIDS (COMPRESSIBLE MIXTURES WITH MICRO-INERTIA)

2.1. Simplified Form of the Governing Equations

We neglect capillary term (5) and virtual kinetic energy (6) in the expression of total energy (1.9). The simplified Lagrangian is

$$L=\sum_{\alpha=1}^{2}\left\{\frac{|\mathbf{j}_\alpha|^2}{2\rho_\alpha}+\frac{m}{2}\left(\frac{d_1\varphi}{dt}\right)^2\delta_{1\alpha}\right\}-\sum_{\alpha=1}^{2}\rho_\alpha\varepsilon_\alpha-\rho e(\varphi). \tag{2.1}$$

Here $\delta_{1\alpha}$ is the Kronecker symbol. We recall that (see formulae (1.4), (1.6), and (1.7))

$$\varphi=\varphi_2,\quad \varphi_1=1-\varphi,\quad \varepsilon_\alpha\left(\rho_\alpha^0,\eta_\alpha\right)=\varepsilon_\alpha\left(\frac{\rho_\alpha}{\varphi_\alpha},\frac{S_\alpha}{\rho_\alpha}\right),$$

$$\rho=\rho_1+\rho_2,\quad \frac{d_1}{dt}=\frac{\partial}{\partial t}+\mathbf{u}_1\nabla=\frac{\partial}{\partial t}+\frac{\mathbf{j}_1\nabla}{\rho_1}.$$

The index "2" will denote the gas phase (bubbles) and "1" the liquid phase. In (2.1) we have supposed that the interface velocity $\mathbf{u}_i$ is equal to the velocity of the liquid phase $\mathbf{u}_1$. This hypothesis means that the pulsation energy

$$\frac{m}{2}\left(\frac{d_1\varphi}{dt}\right)^2$$

is concentrated essentially in the liquid phase. This is eventually valid for small volume fractions of bubbles. However, for large concentrations this choice may be not the best. Nevertheless, we accept this hypothesis, which turns out to be quite good in practice. To simplify

the final formulae we also suppose that $m = m(\varphi)$. In general, $m = m(\varphi, \rho_1, \rho_2, S_1, S_2, |\mathbf{u}_2 - \mathbf{u}_1|)$. This parameter represents the mass of the fluid involved into the motion during the bubble pulsations.

First, we shall give an explicit form of the equations of motion (1.17). We get from (2.1) the following expressions:

$$\mathbf{K}_\alpha = \frac{\partial L}{\partial \mathbf{j}_\alpha} = \mathbf{u}_\alpha + m\frac{d_1\varphi}{dt}\frac{\nabla\varphi}{\rho_1}\delta_{1\alpha},$$

$$R_\alpha = \frac{\partial L}{\partial \rho_\alpha} = -\frac{|\mathbf{u}_\alpha|^2}{2} - m\left(\frac{d_1\varphi}{dt}\right)\frac{\mathbf{u}_1\nabla\varphi}{\rho_1}\delta_{1\alpha} - \frac{\partial}{\partial\rho_\alpha}(\rho_\alpha\varepsilon_\alpha) - e(\varphi)$$

$$\theta_\alpha = -\frac{\partial L}{\partial S_\alpha} = \frac{\partial}{\partial S_\alpha}(\rho_\alpha\varepsilon_\alpha).$$

Hence, Eq. (1.17) is

$$\rho_\alpha\frac{d_\alpha\mathbf{K}_\alpha}{dt} - \left(\frac{\partial\mathbf{K}_\alpha}{\partial\mathbf{x}}\right)^T\mathbf{j}_\alpha - \rho_\alpha\nabla\left(-\frac{|\mathbf{u}_\alpha|^2}{2} - m\left(\frac{d_1\varphi}{dt}\right)\frac{\mathbf{u}_1\nabla\varphi}{\rho_1}\delta_{1\alpha} - \frac{\partial}{\partial\rho_\alpha}(\rho_\alpha\varepsilon_\alpha) - e(\varphi)\right)$$
$$+ S_\alpha\nabla\left(\frac{\partial}{\partial S_\alpha}(\rho_\alpha\varepsilon_\alpha)\right) = 0.$$

The Gibbs identity (1.8) is equivalent to the identity $d(\rho_\alpha\varepsilon_\alpha) = \theta_\alpha dS_\alpha + \mu_\alpha d\rho_\alpha - p_\alpha d\varphi_\alpha$, where

$$\mu_\alpha = \varepsilon_\alpha + \frac{p_\alpha\varphi_\alpha}{\rho_\alpha} - \theta_\alpha\eta_\alpha$$

is the Gibbs potential. It implies that $S_\alpha d\theta_\alpha + \rho_\alpha d\mu_\alpha = \varphi_\alpha dp_\alpha$. Hence, $S_\alpha\nabla\theta_\alpha + \rho_\alpha\nabla\mu_\alpha = \varphi_\alpha\nabla p_\alpha$. We obtain then the momentum equations in the form

$$\rho_\alpha\frac{d_\alpha\mathbf{K}_\alpha}{dt} - \left(\frac{\partial\mathbf{K}_\alpha}{\partial\mathbf{x}}\right)^T\mathbf{j}_\alpha + \left(\frac{\partial\mathbf{u}_\alpha}{\partial\mathbf{x}}\right)^T\mathbf{j}_\alpha + \varphi_\alpha\nabla p_\alpha + \rho_\alpha\frac{de}{d\varphi}\nabla\varphi$$
$$+ \rho_\alpha\nabla\left(m\left(\frac{d_1\varphi}{dt}\right)\frac{\mathbf{u}_1\nabla\varphi}{\rho_1}\right)\delta_{1\alpha} = 0.$$

For $\alpha = 2$ we have $\mathbf{K}_2 = \mathbf{u}_2$, $\delta_{12} = 0$ and hence the momentum equation for the second component is

$$\frac{\partial\rho_2\mathbf{u}_2}{\partial t} + div(\rho_2\mathbf{u}_2\otimes\mathbf{u}_2 + \varphi_2 p_2 I) = \left(p_2 - \rho_2\frac{de}{d\varphi_2}\right)\nabla\varphi. \tag{2.2}$$

For $\alpha = 1$ we get

$$\mathbf{K}_1 = \mathbf{u}_1 + m\frac{d_1\varphi}{dt}\frac{\nabla\varphi}{\rho_1}$$

and the momentum equation will be

$$\frac{\partial\rho_1\mathbf{K}_1}{\partial t} + div(\rho_1\mathbf{u}_1\otimes\mathbf{K}_1 + p_1\varphi_1 I) = -\left(p_1 + \rho_1\frac{de}{d\varphi}\right)\nabla\varphi - m\frac{d_1\varphi}{dt}\left(\frac{\partial\mathbf{u}_1}{\partial\mathbf{x}}\right)^T\nabla\varphi.$$

Or, in terms of the velocity $\mathbf{u}_1$

$$\rho_1 \frac{d_1 \mathbf{u}_1}{dt} + \nabla(p_1\varphi_1) + \rho_1 \frac{d_1}{dt}\left(m\frac{d_1\varphi}{dt}\frac{1}{\rho_1}\right)\nabla\varphi + m\frac{d_1\varphi}{dt}\nabla\left(\frac{d_1\varphi}{dt}\right) = -\left(p_1 + \rho_1\frac{de}{d\varphi}\right)\nabla\varphi. \tag{2.3}$$

Finally, straightforward calculations give the microstructural equation (1.18) for the Lagrangian (2.1) in the following condensed form:

$$\frac{d_1}{dt}\left(\frac{m(\varphi)}{2\rho_1^2}\left(\frac{d_1\varphi}{dt}\right)^2\right) = \left(p_2 - p_1 - \rho\frac{de}{d\varphi}\right)\frac{d_1\varphi}{dt}\frac{1}{\rho_1^2}.$$

If we put

$$\tau = \frac{\frac{d_1\varphi}{dt}\sqrt{m}}{\rho_1},$$

the microstructural equation can be rewritten as the first-order quasi-linear system:

$$\begin{aligned} \frac{d_1\varphi}{dt} &= \frac{\tau\rho_1}{\sqrt{m}} \\ \frac{d_1\tau}{dt} &= \frac{p_2 - p_1 - \rho\frac{de}{d\varphi}}{\rho_1\sqrt{m}}. \end{aligned} \tag{2.4}$$

The variable τ is an analogue of the bubble radial velocity in the Rayleigh–Lamb equation (see (5.1)). The variables φ, τ are the Riemann invariants corresponding to the velocity field $\mathbf{u}_1$. They admit the following divergence form:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_1\varphi) + div(\rho_1\varphi\mathbf{u}_1) &= \frac{\tau\rho_1^2}{\sqrt{m}} \\ \frac{\partial}{\partial t}(\rho_1\tau) + div(\rho_1\tau\mathbf{u}_1) &= \frac{p_2 - p_1 - \rho\frac{de}{d\varphi}}{\sqrt{m}}. \end{aligned} \tag{2.5}$$

Equations (2.4), or their divergence form (2.5), allow us to rewrite the momentum equation (2.3) for the first component in the following form:

$$\rho_1\frac{d_1\mathbf{u}_1}{dt} + \nabla\left(p_1\varphi_1 + \frac{m}{2}\left(\frac{d_1\varphi}{dt}\right)^2\right) = \left(p_2 - \rho_2\frac{de}{d\varphi}\right)\nabla\varphi_1 \equiv -\left(p_2 - \rho_2\frac{de}{d\varphi}\right)\nabla\varphi. \tag{2.6}$$

Equations (2.2), (2.4), and (2.6) supplemented by the equations of conservation of mass and entropy (1.2) and (1.5), form a closed system of governing equations expressed in terms of usual physical variables. We present them together to simplify the study of hyperbolicity. They are

$$\begin{aligned} \frac{\partial\rho_2}{\partial t} + div(\rho_2\mathbf{u}_2) &= 0 \\ \frac{\partial\rho_1}{\partial t} + div(\rho_1\mathbf{u}_1) &= 0 \\ \rho_2\frac{d_2\mathbf{u}_2}{dt} + \nabla(p_2\varphi_2) &= \left(p_2 - \rho_2\frac{de}{d\varphi}\right)\nabla\varphi \equiv p_i\nabla\varphi \end{aligned}$$

$$\rho_1 \frac{d_1 \mathbf{u}_1}{dt} + \nabla\left(p_1\varphi_1 + \frac{\tau^2\rho_1^2}{2}\right) = \left(p_2 - \rho_2\frac{de}{d\varphi}\right)\nabla\varphi_1 \equiv -p_i\nabla\varphi$$

$$\frac{d_1\varphi}{dt} = \frac{\tau\rho_1}{\sqrt{m}}$$

$$\frac{d_1\tau}{dt} = \frac{p_2 - p_1 - \rho\frac{de}{d\varphi}}{\rho_1\sqrt{m}}$$

$$\frac{d_2\eta_2}{dt} = 0$$

$$\frac{d_1\eta_1}{dt} = 0. \tag{2.7}$$

Here the pressure

$$p_i = p_2 - \rho_2\frac{de}{d\varphi}$$

may be interpreted as the average pressure on the interface separating the two phases.

2.2. Hyperbolicity

We will check now the hyperbolicity of system (2.7). For the sake of simplicity we consider the one-dimensional case. Let c_α be the sound velocity of α-th component. We get

$$\nabla p_1 = \frac{\partial p_1}{\partial \rho_1^0}\left(\frac{\nabla\rho_1}{\varphi_1} - \frac{\rho_1}{\varphi_1^2}\nabla\varphi_1\right) + \frac{\partial p_1}{\partial \eta_1}\nabla\eta_1 = c_1^2\frac{\nabla\rho_1}{\varphi_1} + \frac{c_1^2\rho_1^0}{\varphi_1}\nabla\varphi + \frac{\partial p_1}{\partial\eta_1}\nabla\eta_1.$$

The same for the second component

$$\nabla p_2 = \frac{\partial p_2}{\partial \rho_2^0}\left(\frac{\nabla\rho_2}{\varphi_2} - \frac{\rho_2}{\varphi_2^2}\nabla\varphi\right) + \frac{\partial p_2}{\partial \eta_2}\nabla\eta_2 = c_2^2\frac{\nabla\rho_2}{\varphi_2} - \frac{c_2^2\rho_2^0}{\varphi_2}\nabla\varphi + \frac{\partial p_2}{\partial\eta_2}\nabla\eta_2.$$

Let $\mathbf{v} = (\rho_2, \rho_1, u_2, u_1, \varphi, \tau, \eta_2, \eta_1)^T$ be the vector of unknown variables of dimension eight (one-dimensional case). System (2.7) can be rewritten in the form

$$\frac{\partial\mathbf{v}}{\partial t} + A(\mathbf{v})\frac{\partial\mathbf{v}}{\partial x} = \mathbf{f},$$

where

$$\mathbf{f} = \left(0, 0, 0, 0, \tau\rho_1/\sqrt{m}, \left(p_2 - p_1 - \rho\frac{de}{d\varphi}\right)\Big/(\rho_1\sqrt{m}), 0, 0\right)^T$$

and the matrix A is given as

$$A=\begin{pmatrix} u_2 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & \rho_1 & 0 & 0 & 0 & 0 \\ \frac{c_2^2}{\rho_2} & 0 & u_2 & 0 & \frac{-c_2^2\rho_2^0+p_2-p_i}{\rho_2} & 0 & \frac{\partial p_2}{\partial \eta_2}\frac{\varphi_2}{\rho_2} & 0 \\ 0 & \frac{c_1^2}{\rho_1}+\tau^2 & 0 & u_1 & \frac{-p_1+p_i+c_1^2\rho_1^0}{\rho_1} & \rho_1\tau & 0 & \frac{\partial p_1}{\partial \eta_1}\frac{\varphi_1}{\rho_1} \\ 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 \end{pmatrix}.$$

The eigenvalues of A are given by

$$\begin{aligned} \lambda_{1,2,3} &= u_1 \\ \lambda_4 &= u_2 \\ \lambda_{5,6} &= u_1 \pm \sqrt{c_1^2+\rho_1\tau^2} \\ \lambda_{7,8} &= u_2 \pm c_2. \end{aligned} \tag{2.8}$$

All the eigenvalues are real. The multiple eigenvalues correspond to the propagation of entropy waves and concentration waves. It is quite easy to show that the right eigenvectors of A constitute a basis of the space R^8, if $(u_2-u_1)^2 \neq c_2^2$ (for details see Embid and Baer [9] who studied a similar question). Hence, we have proved the following theorem:

THEOREM 2. *System* (2.7) *is hyperbolic, if* $(u_2-u_1)^2 \neq c_2^2$. *Its eigenvalues are calculated in explicit form.*

The last two equations of (2.7) expressing the entropy conservation are not convenient for numerical computations. We have to transform them into the energy equations. The energy equations can be obtained by using the Gibbs identity (1.8) that gives, with the mass and the entropy equations, the following identity:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho_\alpha\varepsilon_\alpha+\frac{|\mathbf{j}_\alpha|^2}{2\rho_\alpha}\right)+div\left(\mathbf{j}_\alpha\left(\frac{|\mathbf{u}_\alpha|^2}{2}+\varepsilon_\alpha+p_\alpha\frac{\varphi_\alpha}{\rho_\alpha}\right)\right) \\ &\quad=\mathbf{u}_\alpha\left(\rho_\alpha\frac{d_\alpha\mathbf{u}_\alpha}{dt}+\nabla(p_\alpha\varphi_\alpha)\right)-p_\alpha\frac{d_\alpha\varphi_\alpha}{dt}. \end{aligned}$$

Using the momentum equations (2.2) and (2.6) and Eq. (2.4), we can transform the entropy equations into the equivalent equations for the energy

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho_2\left(\varepsilon_2+e+\frac{|\mathbf{u}_2|^2}{2}\right)\right)+div\left(\rho_2\mathbf{u}_2\left(\varepsilon_2+e+\frac{|\mathbf{u}_2|^2}{2}+p_2\frac{\varphi_2}{\rho_2}\right)\right) \\ &\quad=-\left(p_2-\rho_2\frac{de}{d\varphi}\right)\frac{\partial\varphi_2}{\partial t}\equiv-p_i\frac{\partial\varphi}{\partial t} \end{aligned} \tag{2.9}$$

$$\frac{\partial}{\partial t}\left(\rho_1\left(\varepsilon_1+e+\frac{\rho_1\tau^2}{2}+\frac{|\mathbf{u}_1|^2}{2}\right)\right)+div\left(\rho_1\mathbf{u}_1\left(\varepsilon_1+e+\rho_1\tau^2+\frac{|\mathbf{u}_1|^2}{2}+p_1\frac{\varphi_1}{\rho_1}\right)\right)$$
$$=-\left(p_2-\rho_2\frac{de}{d\varphi}\right)\frac{\partial\varphi_1}{\partial t}\equiv p_i\frac{\partial\varphi}{\partial t}. \tag{2.10}$$

System (2.7) where the entropy equations is replaced by the energy equations (2.9) and (2.10), is reminiscent of the BN model (1986) (see also Kapila *et al.* [24], Saurel and Abgrall [36], and Bdzil *et al.* [5]). The difference is the presence of "turbulent" terms in the momentum and energy equations for the continuous phase and a different form of the equation for volume fraction. It is no more a transport equation for φ, but a system of two equations for the volume fraction and its velocity. The theory of Passman *et al.* [32] postulates this microstructural equation (see also a discussion in Bdzil *et al.* [5]). Here a microstructural equation is explicitly obtained by using the variational approach. Its implications on the other parts of the system are also obtained.

Another interesting point is the choice of the interface velocity $\mathbf{u}_i$ and the average interface pressure p_i. For a given velocity $\mathbf{u}_i$ the variational principle determines uniquely the interface pressure p_i. For example, for the bubbly fluid the choice $\mathbf{u}_i=\mathbf{u}_1$ implies $p_i=p_2$, if we neglect the configuration energy $e(\varphi)$. In Baer and Nuziato (1986) the same asymmetric duality takes place for the model describing deflagration-to-detonation transition in granular materials.

It is necessary to underline that the variational principle is very sensitive to the choice of the material derivative

$$\frac{d_i}{dt}.$$

If we take $\mathbf{u}_i=\mathbf{u}_2$, it places "turbulent" terms into the equations for the dispersed phase (gas phase) and not into the equations for the continuous phase (liquid phase). In general situations, the choice of the "turbulent energy" in the form (1.22) would be preferable.

3. DISSIPATIVE MODEL

In our applications the "configuration energy" $e(\varphi)$ is neglected. In this case, the interface pressure $p_i=p_2$ and the interface velocity $\mathbf{u}_i=\mathbf{u}_1$. We consider a dissipative model for this particular case. To take into account a drag force between liquid and gas, the bubble damping, and the external force $\mathbf{g}$, only minor modifications should be added into system (2.7). To present the system in a form adopted already in Saurel and Abgrall [36] for numerical calculations, we change the order in which the equations are written. The dissipative system used for numerical computations can be cast in the form:

$$\frac{d_1\tau}{dt}=\frac{p_2-p_1-p_\mu}{\rho_1\sqrt{m}}$$
$$\frac{d_1\varphi}{dt}=\frac{\tau\rho_1}{\sqrt{m}}$$
$$\frac{\partial\rho_2}{\partial t}+div(\rho_2\mathbf{u}_2)=0$$
$$\frac{\partial}{\partial t}(\rho_2\mathbf{u}_2)+div(\rho_2\mathbf{u}_2\otimes\mathbf{u}_2+p_2\varphi_2 I)=p_2\nabla\varphi+\rho_2\mathbf{g}+\lambda(\mathbf{u}_1-\mathbf{u}_2)$$

$$\frac{\partial}{\partial t}\left(\rho_2\left(\varepsilon_2+\frac{|\mathbf{u}_2|^2}{2}\right)\right)+div\left(\rho_2\mathbf{u}_2\left(\varepsilon_2+\frac{|\mathbf{u}_2|^2}{2}+p_2\frac{\varphi_2}{\rho_2}\right)\right)$$
$$=-p_2\left(\frac{\tau\rho_1}{\sqrt{m}}-\mathbf{u}_1\nabla\varphi\right)+\lambda\mathbf{u}_1(\mathbf{u}_1-\mathbf{u}_2)+\rho_2\mathbf{u}_2\mathbf{g}$$
$$\frac{\partial\rho_1}{\partial t}+div(\rho_1\mathbf{u}_1)=0$$
$$\frac{\partial}{\partial t}(\rho_1\mathbf{u}_1)+div\left(\rho_1\mathbf{u}_1\otimes\mathbf{u}_1+\left(\frac{\rho_1^2\tau^2}{2}+p_1\varphi_1\right)I\right)=-p_2\nabla\varphi+\rho_1\mathbf{g}+\lambda(\mathbf{u}_2-\mathbf{u}_1)$$
$$\frac{\partial}{\partial t}\left(\rho_1\left(\varepsilon_1+\frac{\rho_1\tau^2}{2}+\frac{|\mathbf{u}_1|^2}{2}\right)\right)+div\left(\rho_1\mathbf{u}_1\left(\varepsilon_1+\rho_1\tau^2+\frac{|\mathbf{u}_1|^2}{2}+p_1\frac{\varphi_1}{\rho_1}\right)\right)$$
$$=p_2\left(\frac{\tau\rho_1}{\sqrt{m}}-\mathbf{u}_1\nabla\varphi\right)-\lambda\mathbf{u}_1(\mathbf{u}_1-\mathbf{u}_2)+\rho_1\mathbf{u}_1\mathbf{g}. \tag{3.1}$$

Here the term $\lambda(\mathbf{u}_1-\mathbf{u}_2)$ represents the Stokes type drag force, λ is a positive function depending on the local characteristics of each component, and p_μ is the viscous force responsible for bubble damping. The following classical expression of p_μ reads for spherical bubbles (see, for example, Plesset and Prosperetti [33]):

$$p_\mu=\frac{4\mu_l}{R}\frac{dR}{dt}.$$

Here μ_l is the liquid dynamic viscosity, R is the bubble radius related to the gas volume fraction φ and the bubble number density N by the formula $\varphi=\frac{4}{3}\pi R^3N$. By supposing that $N\approx const$ and by replacing $\frac{d}{dt}$ by $\frac{d_1}{dt}$ we can rewrite p_μ in the form

$$p_\mu=\frac{4}{3}\frac{\tau\rho_1\mu_l}{\varphi\sqrt{m}}. \tag{3.2}$$

We will use formula (3.2) even in the case when N is not constant (weak variations). In this case we should add the following equation for N:

$$\frac{\partial N}{\partial t}+div(N\mathbf{u}_2)=0.$$

System (3.1) takes into account mechanical and thermal disequilibrium (we do not suppose that the temperature of the phases are the same). But we do not add in our system relaxation terms corresponding to the thermal disequilibrium.

Now, we have to verify the entropy inequality for system (3.1) (Passman *et al.* [32] and Baer and Nunziato [2]):

$$\sum_{\alpha=1}^{2}\rho_\alpha\frac{d_\alpha\eta_\alpha}{dt}\geq 0. \tag{3.3}$$

Straightforward calculations show that

$$\sum_{\alpha=1}^{2}\rho_\alpha\frac{d_\alpha\eta_\alpha}{dt}=\lambda\frac{(\mathbf{u}_2-\mathbf{u}_1)^2}{\theta_2}+\frac{\tau\rho_1 p_\mu}{\sqrt{m}\theta_1}.$$

Expression (3.2) of p_μ guarantees that the second term is positive. Hence, the entropy inequality (3.3) is verified.

4. NUMERICAL METHODS

In order to deal with the applications involving compressible mixtures, we need two types of numerical methods. The first one is a method able to solve new multiphase model (3.1). The second method is used to solve the classical multidimensional Euler equations in the presence of material interfaces, in order to provide reference solutions for the multiphase model.

4.1. Numerical Method for Multiphase Equations

In Saurel and Abgrall [36] a numerical method was proposed to resolve a nonconservative system describing multiphase flows. We note that the structure of the new model is close to the one proposed in Baer and Nunziato [2] and Saurel and Abgrall [36] (the BN-type model). To show this we rewrite system (3.1) in the one-dimensional case. We shall use the indices "l" and "g" for the local characteristics of the liquid and gas phases. Moreover, we drop the superscript "0" in the notations of the local densities. Hence, we denote

$$u_1 = u_l, \quad u_2 = u_g, \quad \varphi_1 = \varphi_l, \quad \varphi_2 = \varphi_g, \quad \rho_1 = \varphi_l \rho_l \quad \text{and} \quad \rho_2 = \varphi_g \rho_g.$$

We introduce now the κ variable. When this variable is equal to zero, the BN-type model is recovered. When κ is equal to one, we get model (3.1). When the mass and heat transfer are negligible, the both models can be rewritten in the condensed form,

$$\begin{aligned} \frac{\partial \boldsymbol{\chi}}{\partial t} + u_l \frac{\partial \boldsymbol{\chi}}{\partial x} &= \mathbf{S}_A(\boldsymbol{\chi}, \mathbf{U}) \\ \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} &= \mathbf{H}(\mathbf{U}) \frac{\partial \varphi_g}{\partial x} + \mathbf{S}_B(\boldsymbol{\chi}, \mathbf{U}), \end{aligned} \tag{4.1}$$

with:

$$\begin{aligned} &\boldsymbol{\chi} = (\kappa\tau, \varphi_g)^T, \\ &\mathbf{U} = (\varphi_g\rho_g, \varphi_g\rho_g u_g, \varphi_g\rho_g E_g, N, \varphi_1\rho_1, \varphi_l\rho_l u_l, \varphi_l\rho_l E_l)^T, \\ &\mathbf{F} = \big(\varphi_g\rho_g u_g, \varphi_g\rho_g u_g^2 + \varphi_g p_g, u_g(\varphi_g\rho_g E_g + \varphi_g p_g), N u_g, \\ &\qquad \varphi_l\rho_l u_l, \varphi_l\rho_l u_l^2 + \varphi_l p_l + \kappa(\varphi_l\rho_l\tau)^2/2, u_l(\varphi_l\rho_l E_l + \varphi_l p_l + \kappa(\varphi_l\rho_l\tau)^2/2)\big)^T \\ &\mathbf{H} = (0, p_g, p_g u_l, 0, 0, -p_g, -p_g u_l)^T \quad \text{and} \quad E_g = \varepsilon_g + u_g^2/2, \\ &\qquad\qquad E_l = \varepsilon_l + u_l^2/2 + \kappa\varphi_l\rho_l\tau^2/2. \end{aligned}$$

The source vectors $\mathbf{S}_A(\boldsymbol{\chi}, \mathbf{U})$ and $\mathbf{S}_B(\boldsymbol{\chi}, \mathbf{U})$ read

$$\mathbf{S}_A(\boldsymbol{\chi}, \mathbf{U}) = \left(\kappa \frac{p_g - p_l - p_\mu}{\varphi_l\rho_l\sqrt{m}}, \kappa \frac{\tau\varphi_l\rho_l}{\sqrt{m}} + (1-\kappa)\mu(p_g - p_l)\right)^T$$

$$\mathbf{S}_B(\chi,\mathbf{U}) = \Bigg(0, \varphi_g\rho_g g + \lambda(u_l-u_g), -\kappa p_g \frac{\tau\varphi_l\rho_l}{\sqrt{m}} + (1-\kappa)p_g\mu(p_g-p_l) + \lambda u_l(u_l-u_g), 0, 0, \varphi_l\rho_l g - \lambda(u_l-u_g), \kappa p_g \frac{\tau\varphi_l\rho_l}{\sqrt{m}} - (1-\kappa)p_g\mu(p_g-p_l) - \lambda u_l(u_l-u_g)\Bigg)^T.$$

The parameter μ in the BN model controls the rate at which the phase pressures tend toward equilibrium. In model (3.1), this parameter is absent. The corresponding equation is replaced by two quasi-linear equations describing the same tendency to equilibrium. The similarity between the two models permits us to use the same numerical approach as proposed in Saurel and Abgrall [36].

The major idea of the method is that the difference scheme should conserve the flow uniform in pressure and velocity. This idea was proposed earlier by Abgrall [1], Shyue [41], and Saurel and Abgrall [35] for the Euler equations of compressible fluids and was applied for the contact interface problems even in the case when the initial conditions were not uniform in pressure and velocity. Then this idea was adapted in Saurel and Abgrall [36] to the BN-type model in order to determine the numerical approximations of the nonconservative terms

$$\mathbf{H}(\mathbf{U})\frac{\partial\varphi_g}{\partial x}$$

and a nonconservative equation for φ_g. In the model (4.1), a new variable τ appears. We note that the pressure and velocity are uniform if and only if the source terms vanish. In particular, the pressure is uniform if and only if τ is equal to zero. Hence, the terms

$$\mathbf{H}(\mathbf{U})\frac{\partial\varphi_g}{\partial x}$$

and the equations for χ can be treated as in Saurel and Abgrall [36].

The hyperbolic solver is based on the second-order Godunov-type method following the MUSCL strategy [44]. The source terms $\mathbf{S}_A$ and $\mathbf{S}_B$ are treated by a standard splitting procedure [43], which is not detailed here.

The predictor step is done by using the primitive variable formulation in order to satisfy the uniformity condition. The primitive variables vector is defined as $\mathbf{W} = (\tau, \varphi_g, \rho_g, u_g, p_g, \rho_l, u_l, p_l, N)^T$. The primitive variable vector at time t_n and at point x_i is denoted by $\mathbf{W}_i^n$ and the associated slopes by $\delta\mathbf{W}_i^n$. When the flow is uniform in velocity and pressure, the slopes related to these variables are zero. The primitive variables at the cell center and on the right and left sides of the cell at the half time step are given by

$$\mathbf{W}_i^{n+1/2} = \mathbf{W}_i^n - \Delta t/2B\big(\mathbf{W}_i^n\big)\delta\mathbf{W}_i^n; \quad \mathbf{W}_{i+1/2,-}^{n+1/2} = \mathbf{W}_i^{n+1/2} + 1/2\delta\mathbf{W}_i^n;$$
$$\mathbf{W}_{i-1/2,+}^{n+1/2} = \mathbf{W}_i^{n+1/2} - 1/2\delta\mathbf{W}_i^n.$$

Analogous notations are used for the variables $\mathbf{U}$.

Here $B(\mathbf{W}_i^n)$ is the Jacobian matrix of the system and is given by

$$B(\mathbf{W}) = \begin{pmatrix} u_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_g(u_g - u_l)/\varphi_g & u_g & \rho_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_g & 1/\rho_g & 0 & 0 & 0 & 0 \\ 0 & \rho_g c_g^2(u_g - u_1)/\varphi_g & 0 & \rho_g c_g^2 & u_g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_l & \rho_l & 0 & 0 \\ \varphi_l \rho_l \tau & \frac{p_g - p_l}{\varphi_l \rho_l} - \rho_l \tau^2 & 0 & 0 & 0 & \varphi_l \tau^2 & u_l & 1/\rho_l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_l c_l^2 & u_l & 0 \\ 0 & 0 & 0 & N & 0 & 0 & 0 & 0 & u_g \end{pmatrix}.$$

The sound speeds of the gas and liquid phases are denoted by c_g and c_l, respectively.

The unknown vector of conservative variables $\mathbf{U}$ is then updated by the modified Godunov formula:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\big(\mathbf{F}\big(\mathbf{U}_{i+1/2,-}^{n+1/2}, \mathbf{U}_{i+1/2,+}^{n+1/2}\big) - \mathbf{F}\big(\mathbf{U}_{i-1/2,-}^{n+1/2}, \mathbf{U}_{i-1/2,+}^{n+1/2}\big)\big) + \Delta t \mathbf{H}\big(\mathbf{U}_i^{n+1/2}\big)\Delta.$$

The conservative fluxes are obtained from the HLL solver [20]:

$$\begin{aligned} &\mathbf{F}\big(\mathbf{U}_{i+1/2,-}^{n+1/2}, \mathbf{U}_{i+1/2,+}^{n+1/2}\big) \\ &\quad = \frac{S_{i+1/2}^+ \mathbf{F}_{i+1/2,-}^{n+1/2} - S_{i+1/2}^- \mathbf{F}_{i+1/2,+}^{n+1/2} + S_{i+1/2}^+ S_{i+1/2}^- \big(\mathbf{U}_{i+1/2,+}^{n+1/2} - \mathbf{U}_{i+1/2,-}^{n+1/2}\big)}{S_{i+1/2}^+ - S_{i+1/2}^-}. \end{aligned} \quad (4.2)$$

Nonconservative terms $\mathbf{H}(\mathbf{U})\frac{\partial \varphi_g}{\partial x}$ are approximated by a formula consistent with the HLL solver [36]:

$$\Delta = \frac{1}{\Delta x}\left[\frac{S_{i+1/2}^+ \varphi_{g_{i+1/2,-}}^{n+1/2} - S_{i+1/2}^- \varphi_{g_{i+1/2,+}}^{n+1/2}}{S_{i+1/2}^+ - S_{i+1/2}^-} - \frac{S_{i-1/2}^+ \varphi_{g_{i-1/2,-}}^{n+1/2} - S_{i-1/2}^- \varphi_{g_{i-1/2,+}}^{n+1/2}}{S_{i-1/2}^+ - S_{i-1/2}^-}\right].$$

The waves speeds S^+ and S^- in the HLL solver (4.2) are the maximum and minimum eigenvalues of the Jacobian matrix B which coincide of course with the eigenvalues of the matrix A (see (2.8)). The wave speeds are given by

$$S_{i\pm1/2}^+ = \max_k(0, \lambda_{k,i\pm1/2,-}, \lambda_{k,i\pm1/2,+}), \quad S_{i\pm1/2}^- = \min_k(0, \lambda_{k,i\pm1/2,-}, \lambda_{k,i\pm1/2,+}).$$

The unknown vector of nonconservative variables $\boldsymbol{\chi}$ is then updated by the scheme,

$$\boldsymbol{\chi}_i^{n+1} = \boldsymbol{\chi}_i^n - \frac{\Delta t}{\Delta x}\big((u_l\boldsymbol{\chi})_{i+1/2}^{HLL} - (u_l\boldsymbol{\chi})_{i-1/2}^{HLL} + \boldsymbol{\chi}_i^{n+1/2}\big(u_{l\,i+1/2}^{HLL} - u_{l\,i-1/2}^{HLL}\big)\big), \quad (4.3)$$

where $(u_l\boldsymbol{\chi})_{i+1/2}^{HLL}$ is obtained from the HLL flux (4.2), while the velocity $u_{l\,i+1/2}^{HLL}$ is given by

$$u_{l\,i+1/2}^{HLL} = \frac{(\varphi_l \rho_l u_l)_{i+1/2}^{HLL}}{(\varphi_l \rho_l)_{i+1/2}^{HLL}}.$$

Note that formula (4.3) is not the same as in Saurel and Abgrall [36]. This slight modification yields better convergence.

The conservative state variables at the cell boundary are defined again under HLL approximation:

$$\mathbf{U}^{HLL}_{i+1/2}\left(\mathbf{U}^{n+1/2}_{i+1/2,-},\mathbf{U}^{n+1/2}_{i+1/2,+}\right)=\frac{S^{+}_{i+1/2}\mathbf{U}^{n+1/2}_{i+1/2,+}-S^{-}_{i+1/2}\mathbf{U}^{n+1/2}_{i+1/2,-}+\mathbf{F}^{n+1/2}_{i+1/2,-}-\mathbf{F}^{n+1/2}_{i+1/2,+}}{S^{+}_{i+1/2}-S^{-}_{i+1/2}}.$$

4.2. Numerical Method for the 2D Euler Equations in the Presence of Contact Interfaces

In Section 5 we will show the comparison between 1D computations by using the multiphase model (3.1) and 2D numerical experiments simulating the interaction of a shock wave with a bubble of different gas. The aim is to show that the 1D multiphase model is a reasonable reduction of the averaged 2D problem.

To obtain the 2D results we need to solve the Euler equations in a situation involving a 2D contact interface separating two gases. We recall here the basic ingredients of this method [1, 35, 41].

The dynamics of each pure material is governed by the Euler equations:

$$\begin{aligned}\frac{\partial\rho}{\partial t}+div(\rho\mathbf{u})&=0\\ \frac{\partial\rho\mathbf{u}}{\partial t}+div(\rho\mathbf{u}\otimes\mathbf{u}+pI)&=0\\ \frac{\partial\rho E}{\partial t}+div(\rho E\mathbf{u}+p\mathbf{u})&=0.\end{aligned}\tag{4.4}$$

This system is closed by the stiffened gas equation of state: $p=(\gamma-1)\rho\varepsilon-\gamma p_\infty$ with standard notations for unknowns. The parameters of the equation of state γ and p_∞ are different for each material and are discontinuous at the contact interface. The method that follows is specific to this type of equation of state. Generalization of the method to the Mie–Gruneisen–type equation of state is done in Massoni *et al.* [29].

In order to deal with very large deformations, we wish to solve these equations by using a Eulerian method. This poses the well-known problem of artificial diffusion of contact discontinuities. At the contact interface, the density and internal energy is smeared over several mesh points. Since the interface separates media with different equations of state, the computation of pressure, sound speed, and all thermodynamic variables is critical.

The parameters of the equation of state are function of space and time: $\gamma=\gamma(\mathbf{x},t)$ and $p_\infty=p_\infty(\mathbf{x},t)$, $\mathbf{x}=(x,y)^T$. The method consists of a succession of one-dimensional sweeps along each direction [43] over a Cartesian mesh. Consider, for example, a step along the x-direction (the step along the y-direction is analogous). The Euler system (4.4) should be supplemented by several evolution equations [35]. First, two evolution equations for the state parameters are necessary. Second, an evolution equation for the kinetic energy associated to the y-velocity component is necessary. Each one-dimensional step consists in

the resolution of the system:

$$\begin{aligned} \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} &= 0 \\ \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} &= 0, \end{aligned} \tag{4.5}$$

with $\mathbf{U} = (\rho, \rho u, \rho v, \rho E, \rho v^2/2)^T$, $\mathbf{F}(\mathbf{U}) = (\rho u, \rho u^2 + p, \rho uv, u(\rho E + p), \rho u v^2/2)^T$ and

$$\omega = \left(\frac{1}{\gamma - 1}, \frac{\gamma p_\infty}{\gamma - 1}\right)^T.$$

Resolution is based on a second-order Godunov scheme with the same type of predictor step under primitive variables formulation as in Section 4.1. The update of the unknown variables is given by the formulae

$$\omega_i^{n+1} = \omega_i^n - \frac{\Delta t}{\Delta x}\left(u^*_{i+1/2}\omega^*_{i+1/2} - u^*_{i-1/2}\omega^*_{i-1/2} + \omega_i^{n+1/2}\left(u^*_{i+1/2} - u^*_{i-1/2}\right)\right)$$

and

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^*\left(\mathbf{U}^{n+1/2}_{i+1/2,-}, \mathbf{U}^{n+1/2}_{i+1/2,+}\right) - \mathbf{F}^*\left(\mathbf{U}^{n+1/2}_{i-1/2,-}, \mathbf{U}^{n+1/2}_{i-1/2,+}\right)\right),$$

where $\omega^*_{i+1/2}$, $u^*_{i+1/2}$, and $\mathbf{F}^*(\mathbf{U}^{n+1/2}_{i+1/2,-}, \mathbf{U}^{n+1/2}_{i+1/2,+})$ are the state parameters, the velocity in x-direction and the flux, respectively, obtained from the solution of the Riemann problem of system (4.5). The Riemann solver for this system is such that the extra variables $(\omega, v, v^2/2)$ do not change across the right and left waves. This Riemann solver is an obvious extension of a conventional exact Riemann solver for the Euler equations.

5. TEST PROBLEMS

Model (3.1) couples the microscale and macroscale motions in a unique set of hyperbolic PDEs. The first test problem we are interested in will be to put into evidence an important influence of the microscopic motion on the macroscopic one. We consider the shock wave propagation in a bubbly fluid and compare numerical results obtained by using a 1D model (3.1) with experimental ones. The second test problem examines the aptitude of model (3.1) to restore averaged two-dimensional numerical results obtained for the 2D Euler equations for the problem of shock–bubble interaction. This topic is close to the one examined by Glimm *et al.* [15].

5.1. Shock Wave Propagation in Bubbly Fluids

Interesting experimental and theoretical works on the shock wave propagation in bubbly fluids have been done for more than 30 years. Some old experimental works were suffering because of a lack of accuracy in physical measurements and because of the presence of solute gases producing the creation of new bubbles.

The experimental results were also sensible to the initial bubble dispersion. Careful experiments have been recently done in Japan by Kameda *et al.* [23] using a liquid exempt of

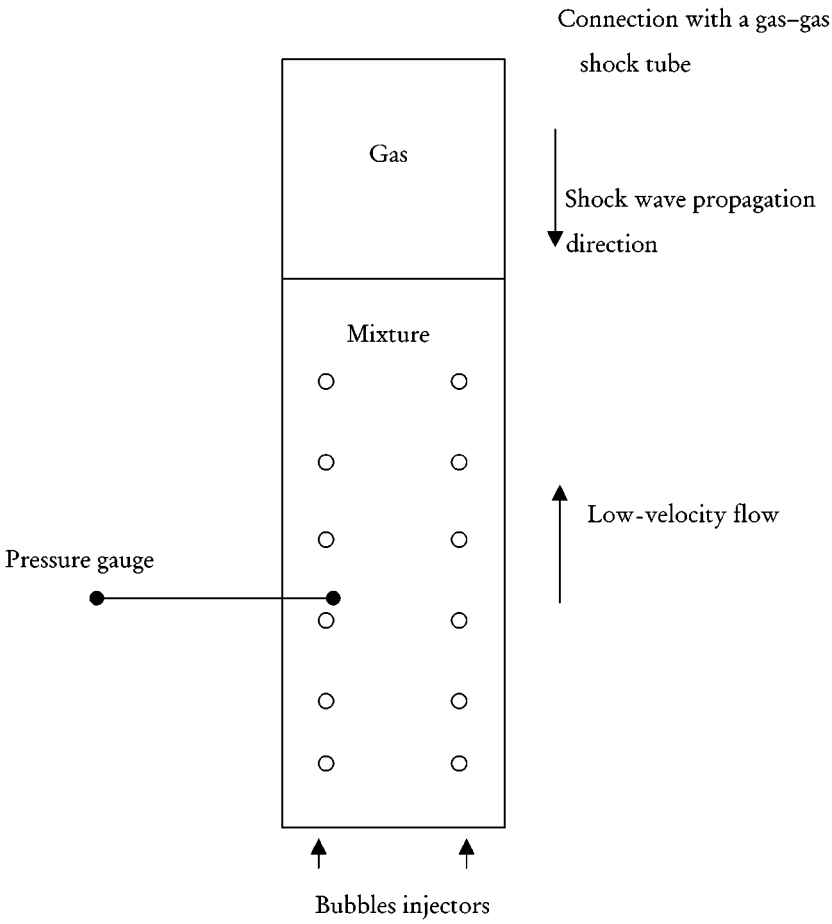

FIG. 1. Schematic representation of the bubbly shock tube.

solute gases. They preserved in their experiments a uniform bubble size-and-space distribution. The experimental apparatus consists of a vertical shock tube. The bubbles are injected on the bottom of the tube and rise by natural and forced convection. A shock wave interacts with the free surface, transmitting a compression wave into the mixture. The experimental situation is shown in Fig 1. A typical record obtained with a pressure gauge placed into the liquid is shown in Fig. 2.

Figure 2 represents the difference between instantaneous pressure and hydrostatic pressure at a given position in the shock tube versus time. The strong coupling between wave dynamics and bubbles pulsation is clearly visible. The experimental conditions and physical data are summarized in Table I.

5.1.1. *Rayleigh–Lamb Test Problem*

This first test is very basic and consists of the following. We consider a tube filled with a bubbly fluid. Both phases are uniform and initially at rest. Only a pressure difference is present between the phases. Thus, each bubble must follow the behavior of an isolated bubble governed by the Rayleigh–Lamb equation [27]:

$$R\ddot{R} + 3/2\dot{R}^2 = (p_g - p_l)/\rho_l. \tag{5.1}$$

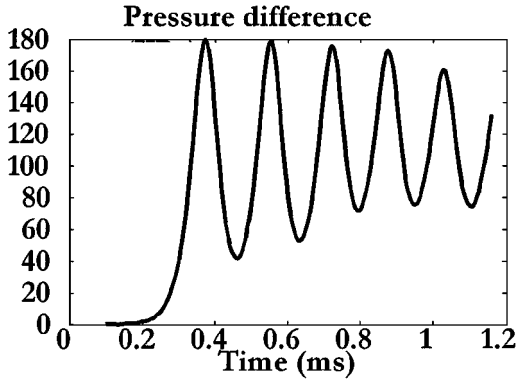

FIG. 2. A typical pressure signal inside the shock tube filled by a bubbly fluid (courtesy of Professor Kameda, University of Tokyo).

This equation is equivalent to (2.4) if we neglect the surface energy $e(\varphi)$ and substitute

$$\rho_1^0 = \rho_l \approx const, \quad \varphi = \frac{4}{3}\pi R^3 N$$

and

$$m(\varphi) = \frac{\rho_l}{3}\left(\frac{3}{4\pi N}\right)^{2/3} \varphi^{-1/3}.$$

The Rayleigh–Lamb equation is easy to solve by assuming that p_l and ρ_l are constant and the gas obeys the polytropic law: $p_g R^{3\gamma} = cst$. In order to compare the results obtained by solving this equation with those obtained by model (3.1), it is necessary assume that:

- All space derivatives disappear. It is what occurs under uniform initial conditions.
- The liquid pressure remains constant. To fulfill this condition, the liquid pressure is imposed to be 2×10^5 Pa instead of using the calculation with equation of state.

TABLE I
Experimental Conditions and Physical Data for the Bubbly Shock Tube

Liquid	Silicone oil
Viscosity	0.00415 (Pa s)
Surface tension	0.00208 (N/m)
Density	953 kg/m3
Sound speed at standard conditions	979 m/s
Stiffened gas equation of state parameters	$\gamma = 3$, $P_\infty = 3.04\ 10^8$ Pa
Gas	SF6
Specific heat ratio	1.09
Molar mass	146 g
Initial temperature	309 K
Initial bubble radius at the location of the pressure gauge	0.613 mm
Pressure gauge location with respect to the free surface	1.462 M
Final pressure jump due to shock wave compression	130.6 KPa

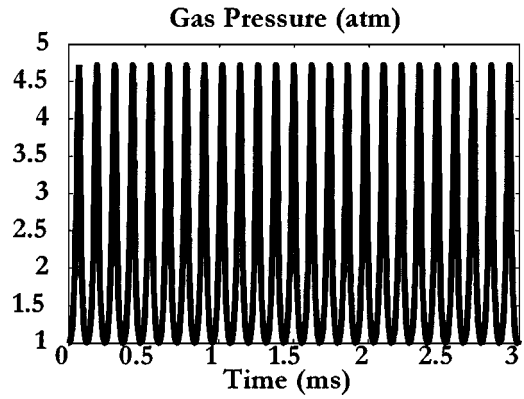

FIG. 3. Comparison between the Rayleigh–Lamb solution and model (3.1) for a uniform bubbly mixture.

The corresponding results are shown in Fig. 3. The two curves and ODE and PDE solutions are superimposed.

5.1.2. *Kameda et al. [23] Test Problem*

Due to the large dilution of the bubbles and of the high-density ratio between fluids, only low sliding between phases is allowed. For this situation it is reasonable to take an infinite velocity relaxation coefficient ($\lambda \to +\infty$). We will adopt for resolution the relaxation procedure detailed in Saurel and Abgrall [36], rather than solving a stiff differential problem. Of course, the model can also be used for finite relaxation coefficient λ.

Instead of simulating a total shock tube with the gas chamber, we consider the impact of a piston with velocity 0.427 m/s, corresponding to the overpressure of 130 KPa as in the Kameda *et al.* [23] experiments, at the distance 1.462 m from the free surface (pressure gauge location). Initial conditions in the tube are shown in Fig. 4 where the gravity effects are visible. The numerical results are shown in Fig. 5 at times 1.92 ms, 3.84 ms, and 5.76 ms. The strong coupling between the macroscopic and microscopic motion (pulsations) is clearly visible.

The graphs of liquid pressure and mixture velocity clearly show the splitting of the initial signal into two waves: low and high-frequency waves. The velocity of the large-amplitude low-frequency wave (5 KHz) is 290 m/s and is close to the gas sound velocity. This is in perfect agreement with the experimental measurement ("about 300 m/s"). The low-amplitude high-frequency wave (20–40 KHz) propagates at the velocity of 827 m/s, which is close to the liquid sound speed.

We conclude this paragraph by a validation with experimental results by Kameda *et al.* [23]. They recorded a pressure signal at a given point of space versus time. We compare here the experimental results shown in Fig. 6 with our numerical results. The number of cells used in the computation is 2500. Experimental results are in bold lines, while numerical ones are in thin lines.

The first three oscillations of large amplitude show a nearly perfect agreement. On the next oscillations, the agreement is a little less accurate but is always inside the experimental error bar of 15 KPa. The precursor wave, predicted by the model (3.1), was not mentioned in the experiments of Kameda *et al.* [23]. However, it was experimentally observed, for example by Kedrinsky [25]. The present results are obtained without any empirical relations.

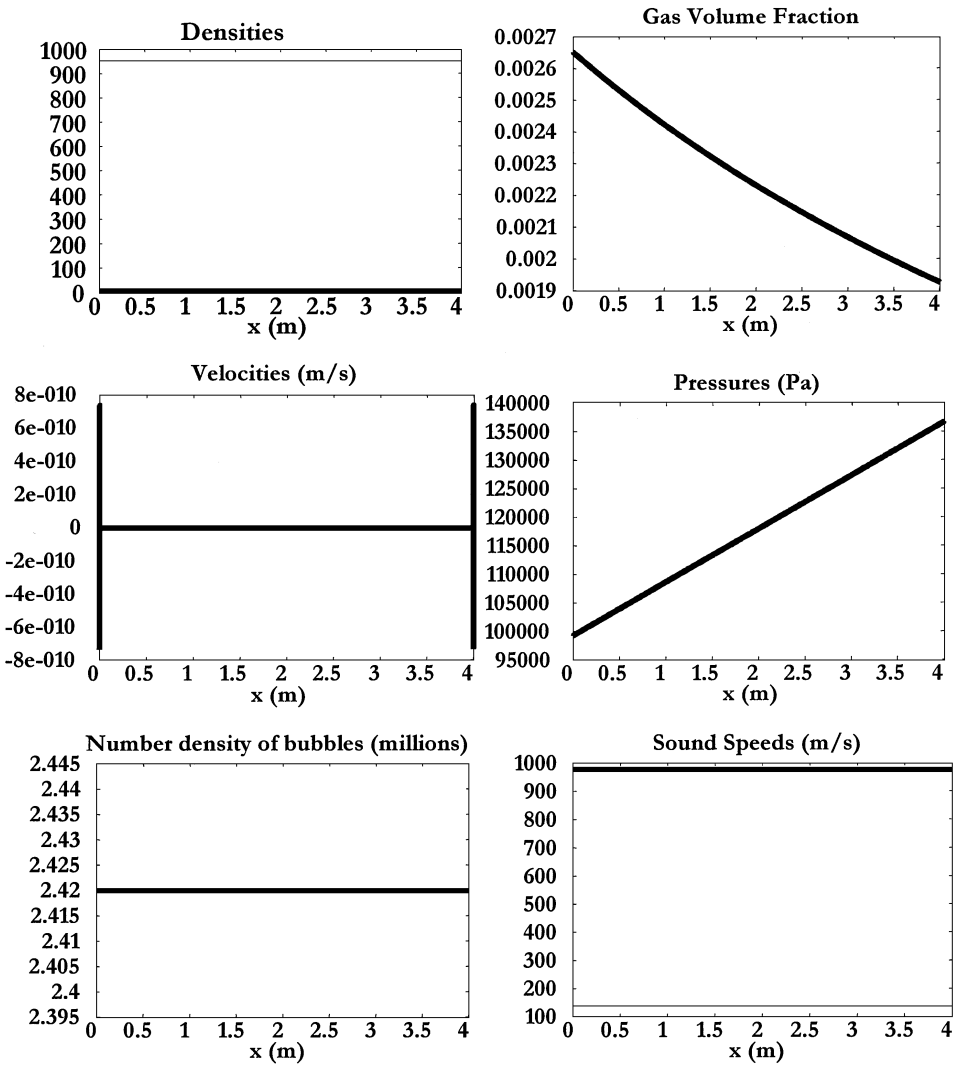

FIG. 4. Initial conditions for the Kameda *et al.* [23] test problem.

5.2. Two-phase One-dimensional Model as a Reduction Tool for Multidimensional Problems

We study the two-dimensional shock wave interaction with a light gas bubble (in fact, a square cylinder) inside a heavy gas. For this configuration we compute a 2D "reference" solution of the Euler equations by using the numerical method described in Section 4.2. We compare the numerical results obtained by using model (3.1) with the averaged 2D "reference" solution.

The physical parameters of the configuration considered are the following. The center of the mass of a square gas cylinder is located at the distance of 2 m on the x-axis. This cylinder is placed inside a shock tube filled with a gas of higher density. The light gas is at rest and has an initial density of 1 kg/m^3 and a pressure of 10^5 Pa. The gas polytropic coefficient is equal to 1.4. The still heavy fluid fills the rest of the domain. The heavy gas

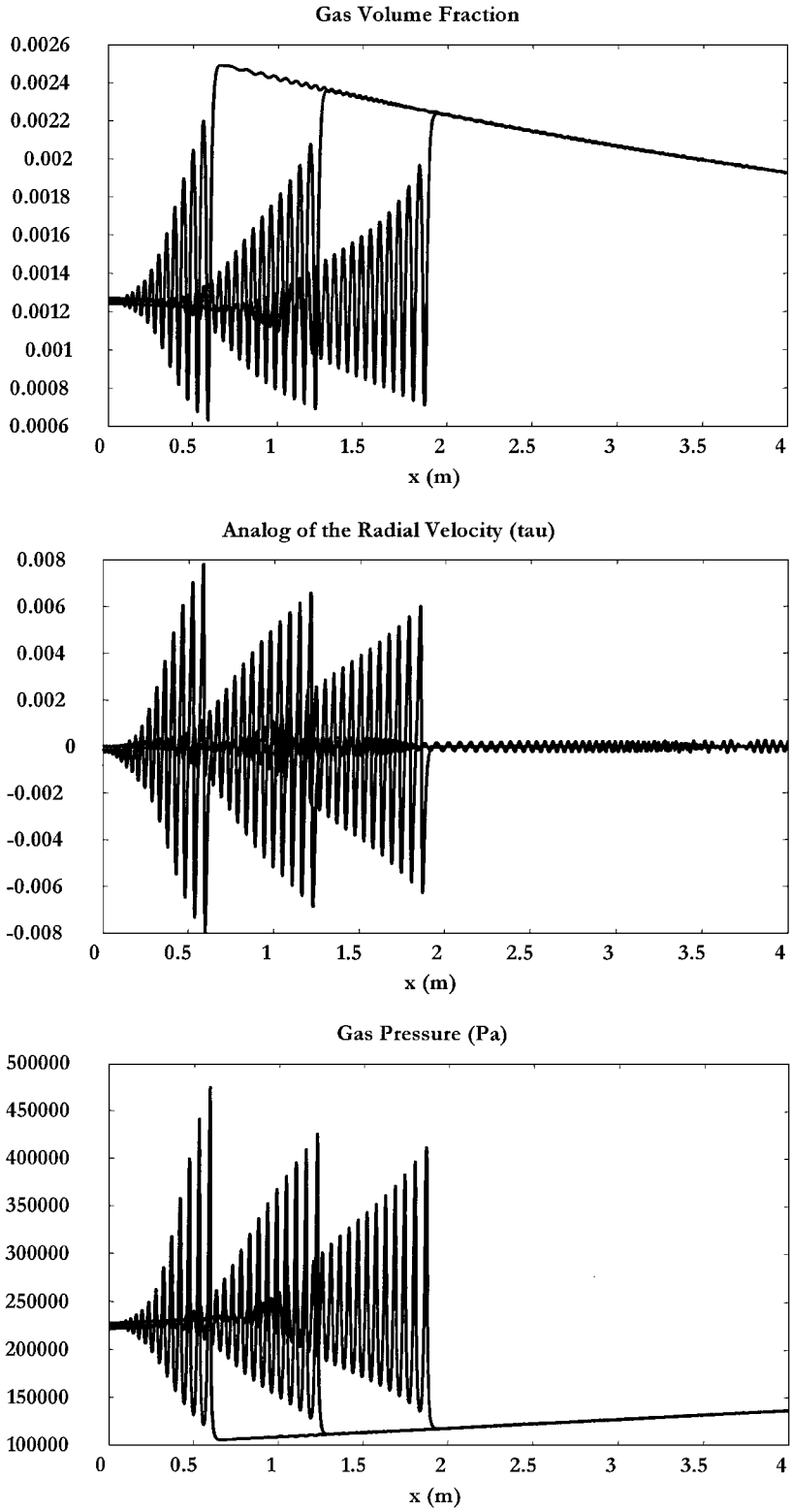

FIG. 5. Two-phase variable profiles obtained by model (3.1) at times 1.92 ms, 3.84 ms and 5.76 ms. Kameda *et al.* [23] test problem.

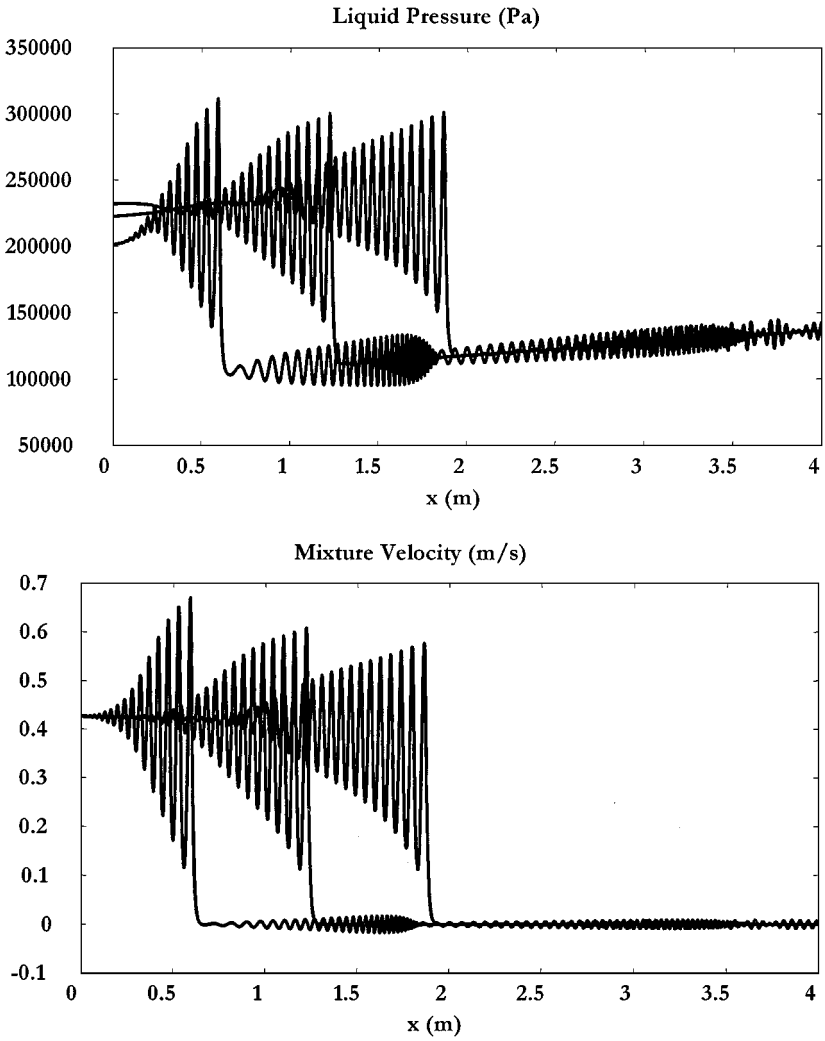

FIG. 5.—*Continued*

polytropic exponent is equal to 3. Its initial density is 10 kg/m^3. The high-pressure (10^6 Pa) chamber begins at $x = 0$ and ends at $x = 1.5$ m. The low-pressure (10^5 Pa) chamber begins at $x = 1.5$ m and ends at $x = 3.5$ m. The total tube width is 0.60 m. This is summarized in Fig. 7.

The 2D evolution of the density field are shown in Fig. 8 at time 1.16 ms, 2.14 ms, 3.13 ms, 4.23 ms, 5.36 ms, 6.50 ms, 7.63 ms, 8.76 ms, 9.89 ms, and 10.09 ms.

The first graph shows the initial stage of the shock interaction with the bubble. In the second graph, bubble deformation begins and the shock wave transmitted forms again. In the third graph, the shock is completely reconstructed. The bubble is now highly deformed. A first jet is clearly visible on the symmetry axis of the domain, producing a bubble rupture. The secondary bubble is compressed again by another Richtmyer–Meshkov instability. From the fourth graph to the end the bubble undergoes an intense rotation. The total number of cells used for the computations is 350 in the x-direction and 30 in the y-direction.

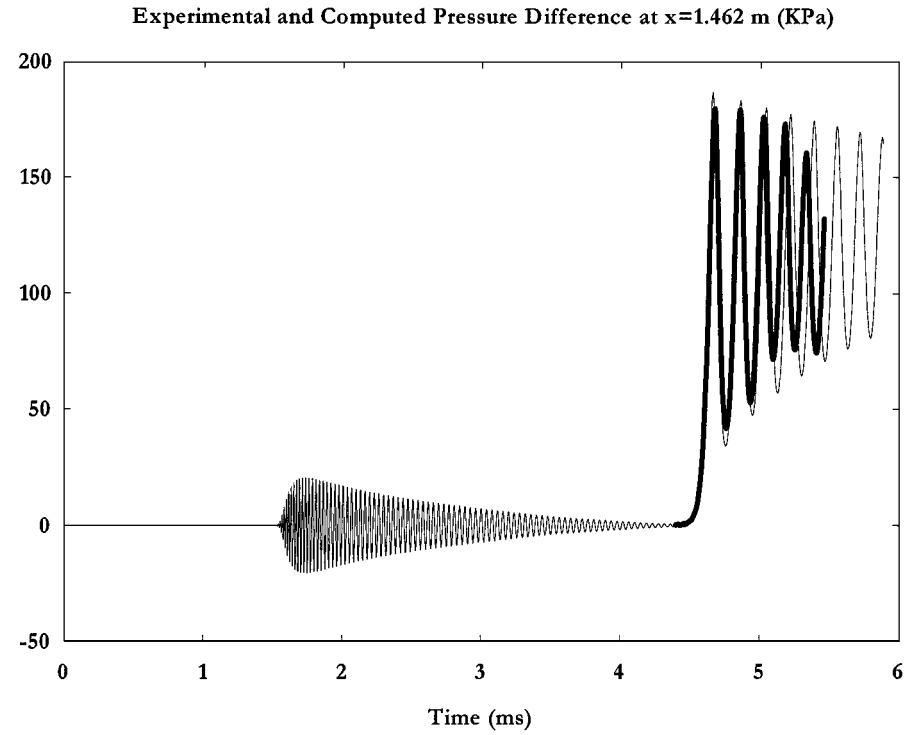

FIG. 6. Comparison of experimental (thick lines) and numerical (thin lines) for the Kameda *et al.* experiments.

Our goal is to compare the averaged two-dimensional results with the 1D two-phase computations. We conserve the same number of cells (350) for 1-D computations. To achieve the 1D multiphase computation, we need closure assumptions. We assume again that the relaxation velocity coefficient λ tends to infinity. We also have to chose the function $m(\varphi_g)$. The function $m(\varphi_g)$ is closely related to the energy stored in the heavy fluid due to inertial phenomena. If $m(\varphi_g)$ is high, the relaxation pressure phenomenon is slow and nonmonotone. When $m(\varphi_g)$ is small, a fast pressure relaxation process occurs. It is not easy to determine this function for compressible fluids and for the present geometry. However, we will justify the hypothesis that for 2D geometry, $m(\varphi_g)$ may be considered as a numerical constant. Indeed, consider a cylinder of radius $R(t)$ in an infinite incompressible fluid. The theory of dimensions shows that the kinetic energy of the fluid per unit length due to the

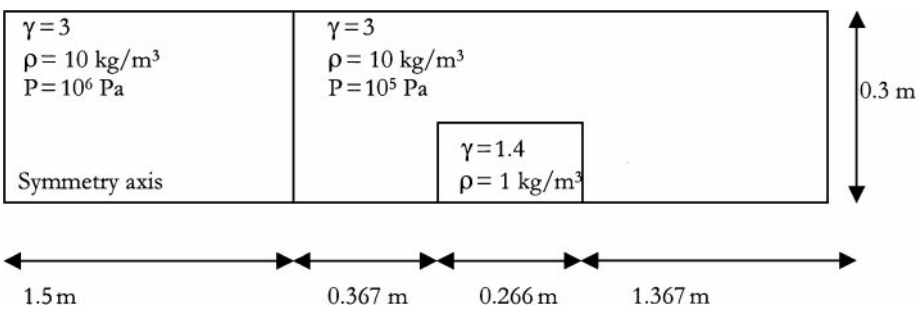

FIG. 7. Numerical experiment configuration for the shock–bubble interaction problem.

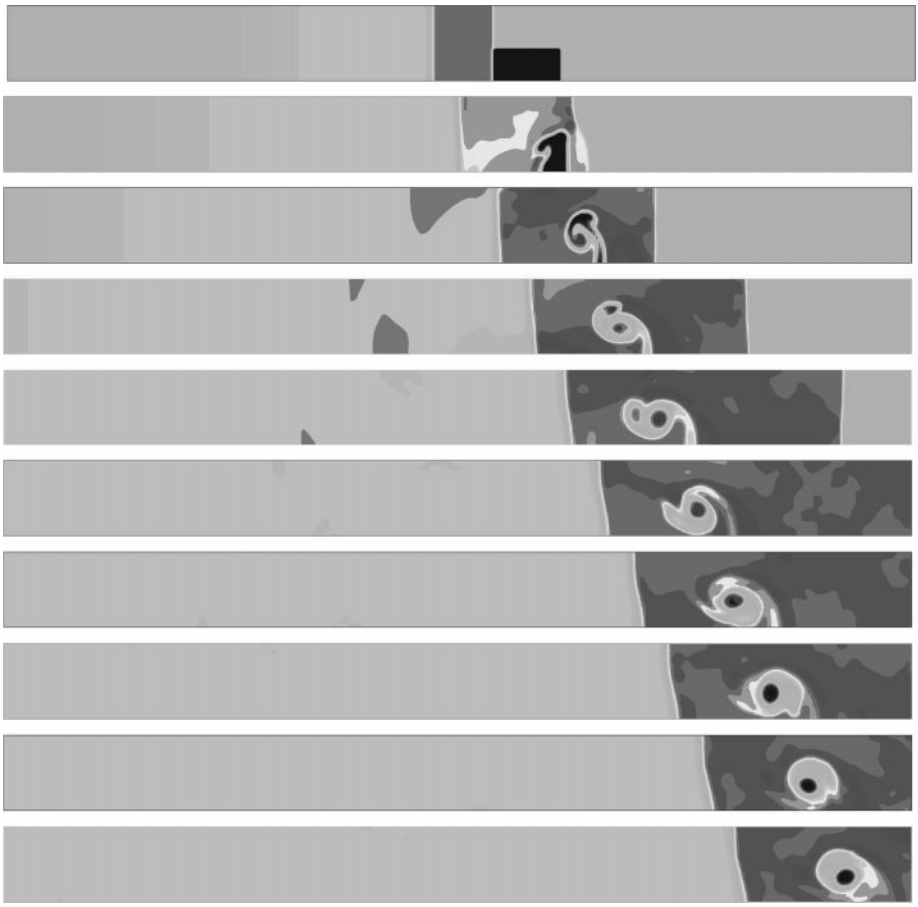

FIG. 8. Density contours at 10 time instants for the shock-bubble interaction problem.

local radial motion of the bubble is then

$$q\rho_1^0 \pi R^2 \left(\frac{dR}{dt}\right)^2,$$

where q is a constant to be determined. In contrast to the spherical case, the explicit value of this constant is unknown. An estimation shows that $q \approx 2$ [25]. Hence, for the 2D case in the presence of N noninteracting cylinders the corresponding kinetic energy per unit length is

$$T_f^c = \frac{q\rho_1^0}{\pi N}\left(\frac{d_i \varphi_g}{dt}\right)^2,$$

where $\varphi_g = \pi R^2 N$ is the volume fraction. This shows that for the plane geometry, $m(\varphi_g)$ does not depend on φ_g and may be considered as a numerical parameter.

We have done a sensitivity analysis of the solution with respect to the parameter m. When this parameter is large (say, greater than 1 ISU), the differences between the solutions of the two models (1D and 2D) are quite strong. When this parameter is smaller than 1, the two solutions agree better. When this function is set to 0.1 or less, the solution agrees quite

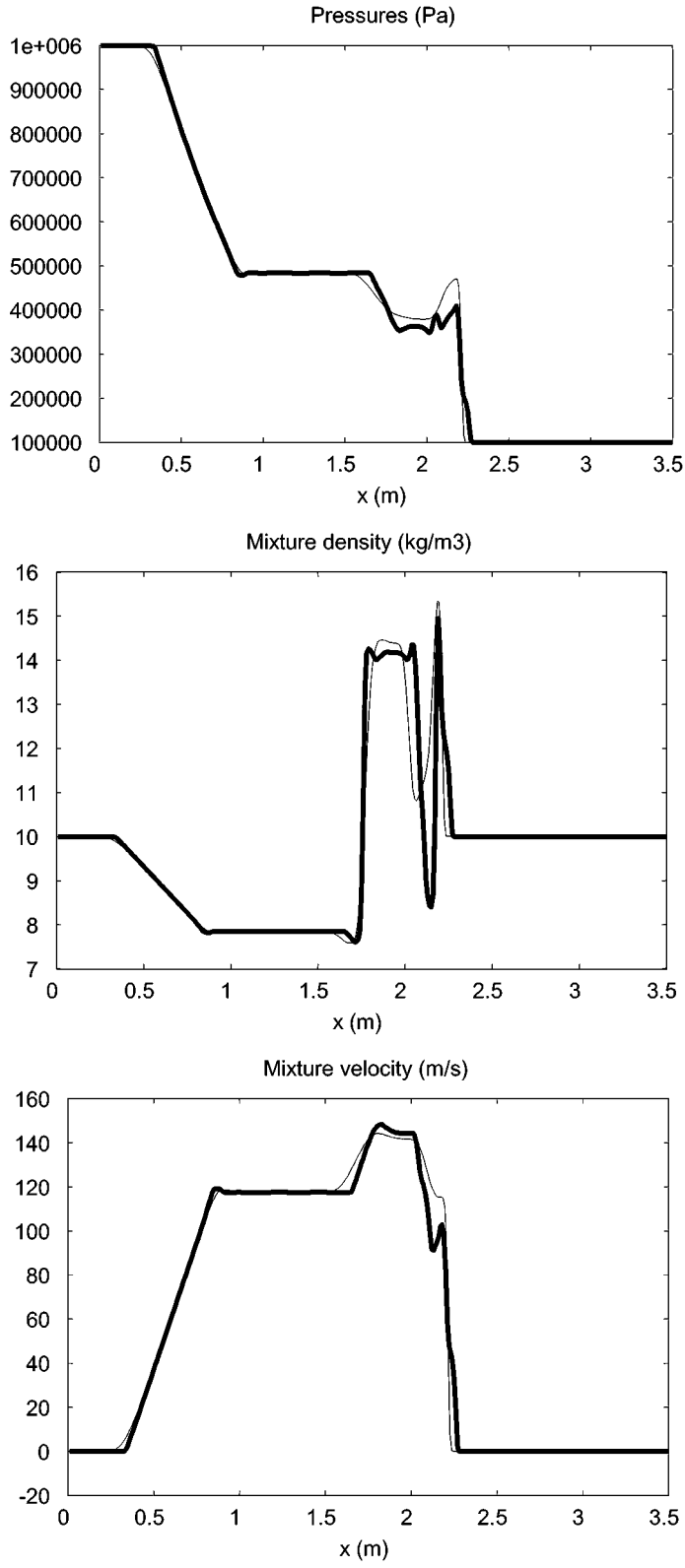

FIG. 9. Comparison of the averaged 2D "reference" solution (bold lines) and 1D solution of two-phase model (3.1) (thin lines) at instant 2.14 ms.

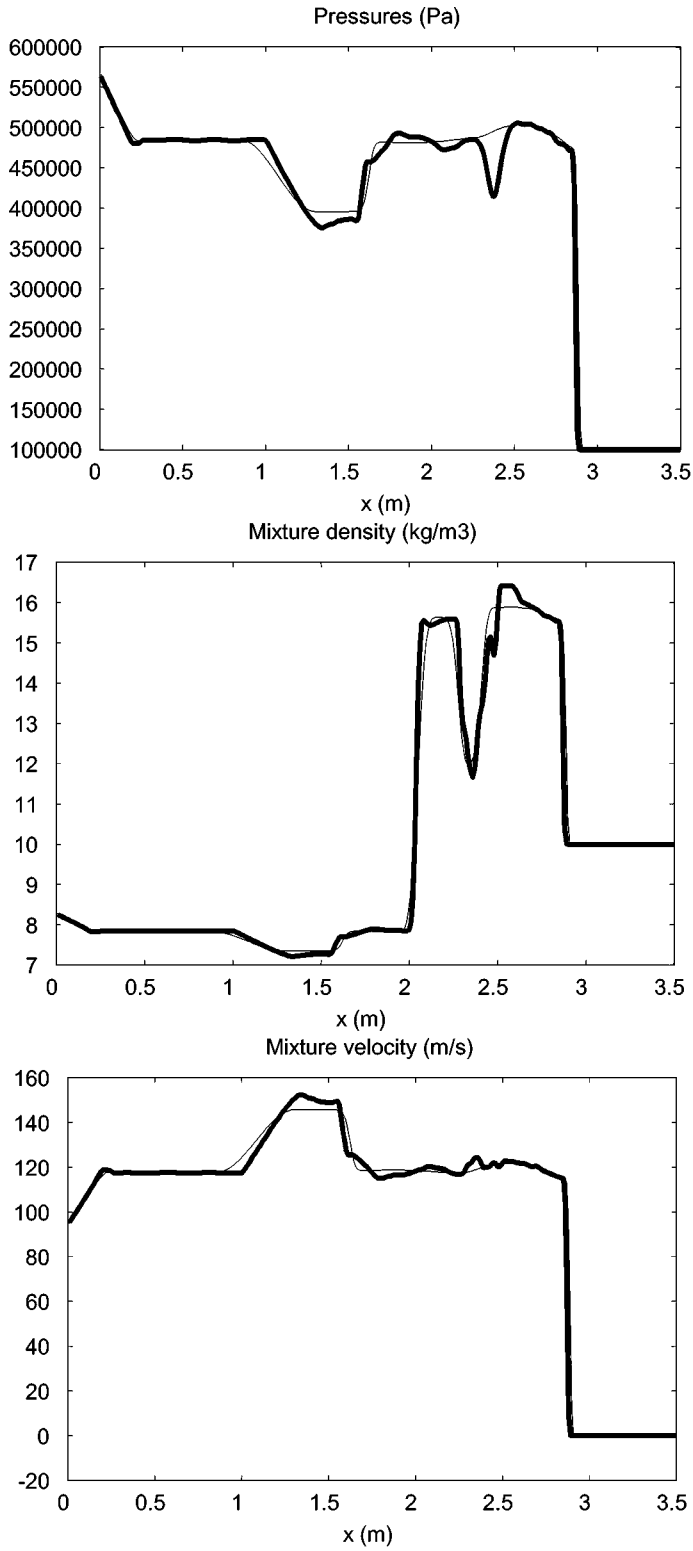

FIG. 10. Comparison of the averaged 2D "reference" solution (bold lines) and 1D solution of two-phase model (3.1) (thin lines) at instant 4.23 ms.

well and is no longer sensitive to this parameter. When it tends to zero, the source terms become stiff, more difficult to solve, but the solution is nearly indiscernible from that with the parameter 0.1. Thus, we retain this value 0.1 for the numerical computations. We have shown in Figs. 9 and 10 at two time instants, the 2D variables averaged over the cross section at each point x (bold lines). In the same figures the results are compared with the 1D multiphase model (3.1) (thin lines).

The comparison of the two solutions is shown at time 2.14 ms (Fig. 9) and 4.23 ms (Fig. 10). On the pressure graph of Fig. 9 the shock wave has reached the bubble and is strongly modified by the interaction. If the bubble was absent the pressure jump would be of 5 atm, where it is 4 atm. The main rarefaction wave travels into the high-pressure chamber and is not sensitive to what occurs in the low-pressure chamber. It is important to note the apparition of a secondary rarefaction wave facing left in the direction of the high-pressure chamber. This rarefaction wave is due to the "bubble" volume compression. Instants later, this wave train is quasi-stationary.

After the interaction with the bubble, the shock wave goes to the pressure value close to the one it would have if the bubble was absent (Fig. 10). More interesting is the pressure oscillation behind the shock front. Its location corresponds perfectly to the bubble position. This pressure decrease is certainly due to the presence of a strong vortex inside the bubble. This indicates that the rotation will be an important phenomenon to consider in the future. The graphs of density and velocity confirm the previous observation concerning pressure behavior.

Comparisons between 2D and 1D calculations show that:

- The main features of the flow are quantitatively preserved by 1D model (3.1) (correct shock wave speed and jumps, main and secondary rarefaction waves),
- The bubble velocity is slightly higher than the velocity of the center of mass obtained by the 1D model (density graph),
- The rotation effects, related to the presence of a vortex inside the bubble, need to be considered in the 1D model.

Even if the agreement is not perfect between the two simulations, these results are very encouraging.

6. CONCLUSION

Compared with existing models, model (3.1) presents important improvements:

- The model takes into account two macroscopic velocities and one microscopic velocity.
- The model does not suppose that the volume fractions are small.
- Compressibility of each phase is taken into account as well as temperature nonequilibrium.

Interesting features of this model are:

- For a particular case of bubbly liquids, the model is hyperbolic apart from a set of parameters of the zero measure. Its eigenvalues are given explicitly. In the general case, a sufficient criterion is proposed: the convexity of the energy implies the hyperbolicity.
- The mathematical structure of the model is very clear. The reason for this clear structure is the use of the Hamilton principle of stationary action and a procedure that is not based on Lagrange multipliers.

This last property is important for the derivation of numerical schemes. We have shown a very simple adaptation of the scheme proposed by Saurel and Abgrall [36].

The new multiphase model has been validated by using experimental data and exact solutions. The next stage will be to take into account the rotation in order to describe all possible types of internal nonequilibrium.

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